

Final mass and spin of black-hole mergers

Wolfgang Tichy and Pedro Marronetti

Department of Physics, Florida Atlantic University, Boca Raton, Florida 33431, USA

(Received 17 July 2008; published 7 October 2008)

We consider black holes resulting from binary black-hole mergers. By fitting to numerical results we construct analytic formulas that predict the mass and spin of the final black hole. Our formulas are valid for arbitrary initial spins and mass ratios and agree well with available numerical simulations. We use our spin formula in the context of two common merger scenarios for supermassive galactic black holes. We consider the case of isotropically distributed initial spin orientations (when no surrounding matter is present) and also the case when matter closely aligns the spins with the orbital angular momentum. The spin magnitude of black holes resulting from successive generations of mergers (with symmetric mass ratio η) has a mean of $1.73\eta + 0.28$ in the isotropic case and 0.94 for the closely aligned case.

DOI: [10.1103/PhysRevD.78.081501](https://doi.org/10.1103/PhysRevD.78.081501)

PACS numbers: 04.25.dg, 04.70.-s, 97.60.Lf

I. INTRODUCTION

Barring the influence of surrounding matter or third objects, two gravitationally bound black holes (BHs) (with masses M_a, M_b and spins \vec{S}_a, \vec{S}_b) will orbit around their common center of mass emitting gravitational radiation, which carries away energy, momentum, and angular momentum. This radiation will circularize the orbits of the progenitor BHs and eventually shrink the orbits until the BHs merge and form a BH of mass M_f and spin S_f . Thus, the initial state is described by eight parameters: the mass ratio $q \equiv M_b/M_a$, the dimensionless spins $\vec{a} = \vec{S}_a/M_a^2$, $\vec{b} = \vec{S}_b/M_b^2$ of the initial BHs, and the dimensionless angular velocity $\omega = \Omega(M_a + M_b)$. Here ω specifies which point on the possibly very long inspiral trajectory is used as the initial point. After the merger the final BH is characterized by seven parameters, the final mass $m = M_f/(M_a + M_b)$, the spin $\vec{s} = \vec{S}_f/M_f^2$, and the kick velocity \vec{k} . Predicting the final m and \vec{s} from the initial parameters is of great importance in many astrophysical merger scenarios. Boyle, Kesden, and Nissanke [1,2] propose to describe any of the final parameters as a Taylor expansion in the six initial spin components. These spin expansions (with coefficients fitted to date from numerical simulations) are based on the assumption that any dimensionless final quantity must be a function of the eight initial parameters. In this paper we use their expansions for the final dimensionless mass and spin. Note, however, that all expansion coefficients still depend on q and ω , since we Taylor expand only in a_i and b_i ; i.e. even if we know the coefficients for a particular mass ratio and a particular initial angular velocity we cannot predict any final quantities for different mass ratios or initial velocities. At first glance this seems to severely limit the usefulness of these expansions. However, as we will see, we have come up with a particular fit for the q dependence, which seems to work rather well. Also, we will show that the final spin magnitude depends only weakly on ω . The orientation of the final spin, however, does depend on ω as one would expect due to spin

precession of the individual BH spins if one starts at a different ω . Nevertheless, our approach can even give approximate spin orientations if we start with values for ω like the ones used here, which are typical for the current state-of-the-art simulations performed by most groups.

We use our formulas to calculate the probability density of the magnitude of the final spin for successive generations of BH mergers. We consider “gas-rich” or “wet” mergers, where a circumbinary disk surrounds the binary, and also “gas-poor” or “dry” mergers, where no matter is present [3,4].

II. SPIN EXPANSIONS

Our coordinate system is such that the z axis is perpendicular to the initial orbital plane. The center of mass is initially at rest at the origin, with the x axis along the direction of the momentum of BH b. The y axis is along the line connecting the BHs with BH a located at $y > 0$.

To construct formulas aimed at predicting the final BH mass and spin, we follow the method described in [2]. To linear order, the spin expansions for the final mass and spin are

$$\begin{aligned} m &= m^0 + (m^{a1}a_z + m^{b1}b_z), \\ s_x &= (s_x^{a1}a_x + s_x^{b1}b_x) + (s_x^{a2}a_y + s_x^{b2}b_y), \\ s_y &= (s_y^{a1}a_x + s_y^{b1}b_x) + (s_y^{a2}a_y + s_y^{b2}b_y), \\ s_z &= s_z^0 + (s_z^{a1}a_z + s_z^{b1}b_z), \end{aligned} \quad (1)$$

where the coefficients are functions of the mass ratio q and the initial orbital angular velocity ω . Notice that by using symmetries such as parity or exchange many terms that would appear in an unconstrained Taylor expansion have been dropped. In addition, all the coefficients enclosed in common brackets are related, by $m^{b1}(q, \omega) = m^{a1}(1/q, \omega)$ and $s_i^{bj}(q, \omega) = s_i^{aj}(1/q, \omega)$.

We ignore the dependence on ω for the time being and focus first on the equal mass case. Note that for $q = 1$ the above mentioned relation between coefficients implies that

TABLE I. Equal mass coefficients up to linear order.

$m^0 = 0.9515 \pm 0.001$	$m^{a1} = m^{b1} = -0.013 \pm 0.007$
$s_x^{a1} = s_x^{b1} = 0.187 \pm 0.002 = +s_y^{a2} = +s_x^{b2}$	
$s_y^{a1} = s_x^{b1} = 0.028 \pm 0.002 = -s_x^{a2} = -s_x^{b2}$	
$s_z^0 = 0.686 \pm 0.004$	$s_z^{a1} = s_z^{b1} = 0.15 \pm 0.03$

all terms enclosed in common brackets have equal coefficients. In order to determine these coefficients we have performed 10 numerical simulations of equal mass binaries with spins of magnitudes between 0.1 and 0.27 with orientations as in Table XI of [2]. All 10 simulations start with $\omega = 0.05$ and are performed using the “moving punctures” method [5,6] with the BAM code [7,8] which allows us to use moving nested refinement boxes. We use 10 levels of 2:1 refinements. The outer boundaries are located $436M$ away from the initial center of mass (M being the sum of the initial BH masses), and our resolution ranges between $8M$ on the outermost box to $M/64$ near the BHs. Apart from the use of sixth order stencils in the interior of the

boxes [9], our setup and methods to determine spin and mass are very similar to the simulations reported in [10–12]. Our 10 simulations are in principle sufficient to determine all coefficients appearing in s_i and m up to quadratic order. However, since the numerical errors in determining the final mass and spin are 0.1% and 0.5%, respectively, and since all quadratic coefficients are small, the errors in all quadratic coefficients are more than 100%. For this reason only the linear coefficients are listed in Table I. In order to verify that linear expansions with the coefficients in Table I give reasonable results we have performed 10 more equal mass runs with spins of magnitude 0.75 with arbitrary orientations. As one can see from the first 10 lines in Table II, the agreement between our additional runs and the values predicted by the expansions is quite good. This demonstrates that for $q = 1$ we can trust the expansions with our coefficients. As we can see from Table I, $s_x^{a1} = s_y^{a2}$ and $s_x^{a2} = -s_y^{a1}$, which is not predicted by the symmetries used to derive the spin expansions. Thus, in terms of the initial spins \vec{S}_a , \vec{S}_b and final spin

TABLE II. Test simulations: the columns show the initial mass ratio q , the progenitor spins a_i and b_i , the final spin magnitude s and mass m , and our predictions s_p and m_p .

q	a_x	a_y	a_z	b_x	b_y	b_z	s	s_p	m	m_p
1:1	.750	.000	-.015	.750	.000	-.015	.772	.755	.949	.952
1:1	.739	-.128	.000	.095	-.680	.303	.777	.773	.945	.948
1:1	-.721	-.061	.199	.280	-.159	-.677	.622	.621	.956	.958
1:1	-.146	-.704	-.215	-.413	-.333	.530	.778	.776	.943	.947
1:1	-.639	-.227	.321	-.517	-.543	.020	.795	.796	.939	.947
1:1	-.431	.340	-.511	.187	-.588	-.426	.542	.544	.960	.964
1:1	-.443	.323	.512	.390	.324	.553	.849	.849	.929	.938
1:1	.136	.405	-.616	.082	-.704	.244	.635	.634	.955	.956
1:1	.010	.187	.726	.174	-.489	.542	.867	.867	.925	.935
1:1	.006	.054	-.748	.398	-.635	-.026	.587	.585	.958	.962
5:6	.000	.000	.000	.000	.000	.000	.682	.682	.952	.952
5:6	.200	.000	-.001	.000	.000	.000	.684	.684	.952	.952
5:6	.000	.150	-.000	.000	.000	.000	.683	.683	.952	.952
5:6	.000	.000	.000	.200	.000	-.000	.683	.683	.952	.952
5:6	.000	.000	.000	.000	.150	-.000	.682	.682	.952	.952
5:6	.200	.000	.099	.000	.000	.000	.703	.702	.950	.951
5:6	.000	.150	-.100	.000	.000	.000	.664	.664	.953	.953
5:6	.000	.000	.000	.200	.000	.100	.695	.695	.951	.951
5:6	.000	.000	.000	.000	.150	-.100	.671	.670	.953	.953
5:6	.000	.150	-.101	.000	.150	-.001	.667	.666	.953	.953
5:6	.200	.000	-.001	-.000	.150	-.100	.673	.672	.953	.953
5:6	.000	.150	-.000	.200	.000	.100	.696	.696	.951	.951
5:6	.200	.150	-.001	.000	.000	.000	.685	.685	.952	.952
5:6	.000	.000	.000	.200	.150	-.001	.683	.683	.952	.952
5:6	.000	.000	.100	.000	.000	.100	.712	.712	.949	.949
5:8	.091	.268	-.412	.043	-.375	.132	.565	.567	.961	.957
5:8	-.287	.225	-.342	.100	-.315	-.226	.556	.558	.962	.961
2:3	-.384	-.033	.106	.112	-.064	-.271	.676	.674	.955	.955
2:3	-.339	-.120	.174	-.207	-.217	.011	.724	.723	.950	.951
2:3	-.230	.180	-.273	.075	-.236	-.169	.587	.588	.960	.959
2:3	.073	.214	-.330	.033	-.281	.099	.596	.597	.959	.956

\vec{S}_f , the x and y components of Eq. (1) can be rewritten as

$$\begin{pmatrix} S_{f,x} \\ S_{f,y} \end{pmatrix} = \alpha R \begin{pmatrix} S_{a,x} + S_{b,x} \\ S_{a,y} + S_{b,y} \end{pmatrix} \quad (2)$$

with

$$R = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}, \quad \beta = 0.15, \quad \alpha = 0.685. \quad (3)$$

This means that the final spin components in the xy -plane are given by rotating and scaling the initial spins. The scale factor is $\alpha = 0.685$ and the rotation matrix is R . This result is physically reasonable, because we know that the initial spins will precess and also radiate some angular momentum. If Eq. (2) also held for unequal masses it would imply a certain form for the mass ratio dependence:

$$\begin{pmatrix} s_x \\ s_y \end{pmatrix} = \alpha R \begin{pmatrix} \frac{M_a^2}{M_f^2} a_x + \frac{M_b^2}{M_f^2} b_x \\ \frac{M_a^2}{M_f^2} a_y + \frac{M_b^2}{M_f^2} b_y \end{pmatrix} \approx \alpha R \begin{pmatrix} \frac{a_x}{(1+q)^2} + \frac{q^2 b_x}{(1+q)^2} \\ \frac{a_y}{(1+q)^2} + \frac{q^2 b_y}{(1+q)^2} \end{pmatrix}. \quad (4)$$

However, the latter cannot be true for extreme mass ratios, since then we would not get $\vec{s} = \vec{a}$ for $q = 0$. In order to get the extreme mass ratio limit correctly, we modify the mass ratio dependence somewhat and assume that

$$\begin{pmatrix} s_x \\ s_y \end{pmatrix} = \begin{pmatrix} s_x^{a1} [g(q, c) a_x + g(1/q, c) b_x] - s_y^{a1} 4\eta (a_y + b_y) \\ s_y^{a1} 4\eta (a_x + b_x) + s_x^{a1} [g(q, c) a_y + g(1/q, c) b_y] \end{pmatrix}, \quad (5)$$

where

$$\eta = \frac{q}{(1+q)^2}, \quad g(q, c) = \frac{(c+1)^2}{(c+q)^2}, \quad (6)$$

$$c = \frac{\sqrt{s_x^{a1}}}{1 - \sqrt{s_x^{a1}}} = 0.762$$

are chosen such that $\vec{s} = \vec{a}$ for $q = 0$ and $\vec{s} = \vec{b}$ for $q = \infty$.

So far we have only considered the components in the xy -plane. For equal masses $s_z^{a1} = s_z^{b1}$ so that s_z in Eq. (1) simplifies. As above we introduce a q dependence given by

$$s_z = s_z^0 (4w\eta + 16(1-w)\eta^2) + s_z^{a1} [g(q, c_3) a_z + g(1/q, c_3) b_z], \quad (7)$$

where $c_3 = \sqrt{s_z^{a1}} / (1 - \sqrt{s_z^{a1}}) = 0.632$. The η dependence of the leading coefficient is inspired by post-Newtonian (PN) expressions, and has been shown to fit runs without initial spins very well [13]. The constant $w = 1.26$ is fitted using the results for unequal masses discussed below.

Similarly, we build a mass ratio dependence into the final mass formula and write

$$m = 1 + (m^0 - 1)4\eta + m^{a1} 16\eta^2 (a_z + b_z). \quad (8)$$

The leading term is inspired by the fact that the binding energy is proportional to η , and the linear term is chosen such that it has the correct limit in the extreme mass ratio case.

While the q dependence we have introduced so far reproduces our fits for the equal mass case and simultaneously gives the correct answers in the extreme mass ratio cases, it is not clear how well our formulas perform for intermediate mass ratios. For this reason we have performed 21 more simulations with q different from unity. As one can see from Table II the predictions are at most 0.4% from the numerical results.

So far we have only kept linear terms in our expansions. We find that our formulas agree with a wide range of test runs. Yet, for initial spin magnitudes close to 1 our formulas deviate from the extrapolated values [12,14,15] for the minimum and maximum possible final spin. These problems can be fixed if we add some quadratic terms to s_z alone:

$$s_z = s_z^{(0)} (4w\eta + 16(1-w)\eta^2) + s_z^{a1} [g(q, c_3) a_z + g(1/q, c_3) b_z] + 16k\eta^2 [(a_x + b_x)^2 + (a_y + b_y)^2 - (a_z + b_z)^2]. \quad (9)$$

These quadratic terms approximately reproduce most of the very small and very uncertain quadratic coefficients in s_z . The coefficient $k = 0.008$ is chosen such that we get the best overall agreement with our numerical simulations.

III. RESULTS

Our particular mapping of initial masses and spins into the final mass and spin given by Eqs. (5), (6), (8), and (9) was fitted for the initial orbital angular velocity $\omega = 0.05$. If we start with the same initial spin components but at lower initial angular velocity (i.e. larger separation) the individual spins and the orbital plane will have precessed by the time we reach $\omega = 0.05$. In this case we cannot expect that our formulas will predict the final spin components if we simply use the initial spin components. However, the final spin magnitude should still be approximately correct, since PN calculations [16] demonstrate that the spin magnitudes are conserved at 2PN order. This expectation is borne out for the following additional $q = 1$ test run. It starts with $\omega = 0.03$, $\vec{a} = (-0.637, -0.226, 0.325)$, $\vec{b} = (-0.517, -0.543, 0.025)$, and yields a final spin $\vec{s} = (-0.226, -0.146, 0.746)$ with magnitude $s = 0.793$. Our fitting formula predicts $\vec{s} = (-0.194, -0.176, 0.753)$ with magnitude $s = 0.797$. This shows that the predicted magnitude is correct up to an error of 0.6%, while the components only agree if the predicted vector is rotated. This rotation comes from the precession during the time it takes to go from $\omega = 0.03$ to $\omega = 0.05$, during which the system completes about 4 orbits.

We have also compared with the numerical results published in [11–13,15,17–20], which all start from an ω that

is not too far from 0.05. The average deviation of all these results from our predictions is about 1%. Hence our formulas give useful predictions for the final mass and even for the spin components if one starts near $\omega = 0.05$. However, if one is ever interested in the final spin components for a much larger initial separation with $\omega \ll 0.05$, one can still evolve the spins using PN theory up to $\omega = 0.05$ and then use our formula. This eliminates the need for expensive numerical simulations.

Other groups have presented formulas that, like ours, attempt to predict the final spin of the merger. The analytic estimate of Buonanno *et al.* [21] can give the final spin magnitude to within a few percent with larger deviations for spins close to anti-alignment. Starting from a number of assumptions, Rezzolla *et al.* [17] developed a more accurate formula with coefficients fitted to numerical results. One of their assumptions is that the components of the final spin in the initial orbital plane are obtained by summing the components of the initial spins i.e., $S_{f,x/y} = S_{a,x/y} + S_{b,x/y}$. In our simulations we observe that this assumption is violated. From the discussion around Eqs. (2) and (3) it is clear that these components get slightly rotated and shortened by a factor $\alpha = 0.685$ during the merger. For instance, the run from the first line in Table II we find a value of $s_{xy} = 0.291$ for the in-plane component of the spin. Our formula (5) predicts a close value of $s_{xy,p} = 0.284$, the approach in [17] leads to $s_{xy,R} = 0.375$ which is about 30% too large, while both approaches predict almost the same spin magnitude. Thus our formulas improve the final spin orientation.

IV. BLACK-HOLE MERGERS

Using Eqs. (5), (6), and (9) we can study the properties of the spin of BHs produced by successive binary BH mergers. Two types of scenarios are likely to witness mergers: one in which the two BHs carry with them dense

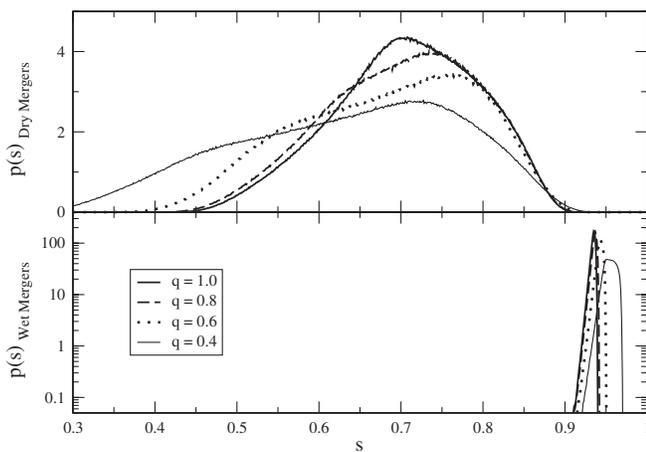


FIG. 1. Probability density (for different mass ratios) of the final BH spin magnitude after 4 generations of dry mergers (top) and wet mergers (bottom).

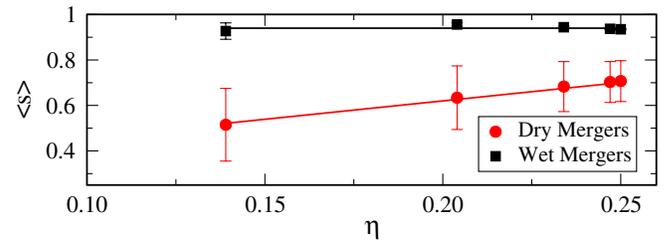


FIG. 2 (color online). Mean value of the final spin magnitude as a function of the symmetric mass ratio η . The vertical bars are the corresponding standard deviations.

surrounding matter (wet mergers) and another where the progenitors meet in relatively empty space (dry mergers) [3]. In the former case the progenitors spins are likely to reach the merger aligned with the orbital angular momentum [22], while in the latter the same spins are bound to be isotropically oriented [3]. In our models we assume that the initial spin directions either have a uniform probability distribution (dry mergers) or that the longitudinal angle θ obeys a normal distribution centered in the direction of the orbital angular momentum (wet mergers) [23]. For the spin magnitude we assume that the original progenitors' spin magnitude is uniformly distributed in the interval $[0,1]$. The probability density for the final spin magnitude after merger can then be obtained with Monte Carlo simulations. Note that the ω dependence of our formula is removed by the integration over the spin orientation. This procedure can be iterated for successive generations assuming that the magnitude of the progenitors spins has the probability density of the previous generation. The resulting probability distribution converges quickly to the curves shown in Fig. 1 for the mass ratios $q = 0.4, 0.6, 0.8$, and 1.0 . The mean values for the probability densities as a function of the symmetric mass ratio η are shown in Fig. 2 with the vertical bars corresponding to the standard deviations. The straight lines are linear fits of the data points. We find that successive dry mergers produce final spins with a mean value of $\langle s \rangle_{\text{Dry}} = 1.73\eta + 0.28$. For $q = 1$ the final spin does not exceed $s = 0.954$ (as our formula predicts for maximal aligned spins). Successive wet mergers produce a spin around 0.94, with very little spread. Similar results have also been found by Berti *et al.* [4] using the spin formula in [17]. This agreement is expected since our spin formulas give magnitudes in close agreement with [17]. The main difference is in the spin orientation which has been integrated out.

V. DISCUSSION

By fitting to numerical results we construct formulas [Eqs. (5), (8), and (9)] that predict the mass and spin of the final BH coming from binary BH mergers. We use them to determine the probability distribution of the final spin magnitude (Figs. 1 and 2) after several generations of mergers of either dry or wet mergers.

ACKNOWLEDGMENTS

It is a pleasure to thank B. Brügmann, L. Rezzolla, E. Barausse, L. Boyle, and M. Kesden for useful discussions. This work was supported by NSF Grant No. PHY-

0652874. We acknowledge TACC at UT Austin for providing HPC resources under allocations No. TG-PHY080022N and No. TG-MCA08X010.

-
- [1] L. Boyle, M. Kesden, and S. Nissanke, *Phys. Rev. Lett.* **100**, 151101 (2008).
 - [2] L. Boyle and M. Kesden, *Phys. Rev. D* **78**, 024017 (2008).
 - [3] T. Bogdanovic, C. S. Reynolds, and M. C. Miller, arXiv: astro-ph/0703054.
 - [4] E. Berti and M. Volonteri, arXiv:0802.0025.
 - [5] M. Campanelli, C. O. Lousto, P. Marronetti, and Y. Zlochower, *Phys. Rev. Lett.* **96**, 111101 (2006).
 - [6] J. G. Baker, J. Centrella, D.-I. Choi, M. Koppitz, and J. van Meter, *Phys. Rev. Lett.* **96**, 111102 (2006).
 - [7] B. Bruegmann, W. Tichy, and N. Jansen, *Phys. Rev. Lett.* **92**, 211101 (2004).
 - [8] B. Bruegmann, J. Gonzalez, M. Hannam, S. Husa, U. Sperhake, and W. Tichy, *Phys. Rev. D* **77**, 024027 (2008).
 - [9] S. Husa, J. A. Gonzalez, M. Hannam, B. Brugmann, and U. Sperhake, *Classical Quantum Gravity* **25**, 105006 (2008).
 - [10] P. Marronetti, W. Tichy, B. Bruegmann, J. Gonzalez, M. Hannam, S. Husa, and U. Sperhake, *Classical Quantum Gravity* **24**, S43 (2007).
 - [11] W. Tichy and P. Marronetti, *Phys. Rev. D* **76**, 061502 (2007).
 - [12] P. Marronetti, W. Tichy, B. Brugmann, J. Gonzalez, and U. Sperhake, *Phys. Rev. D* **77**, 064010 (2008).
 - [13] E. Berti *et al.*, *Phys. Rev. D* **76**, 064034 (2007).
 - [14] M. Campanelli, C. O. Lousto, Y. Zlochower, B. Krishnan, and D. Merritt, *Phys. Rev. D* **75**, 064030 (2007).
 - [15] L. Rezzolla *et al.*, *Astrophys. J.* **679**, 1422 (2008).
 - [16] L. E. Kidder, *Phys. Rev. D* **52**, 821 (1995).
 - [17] L. Rezzolla *et al.*, *Phys. Rev. D* **78**, 044002 (2008).
 - [18] S. Dain, C. O. Lousto, and Y. Zlochower, *Phys. Rev. D* **78**, 024039 (2008).
 - [19] C. O. Lousto and Y. Zlochower, arXiv:0805.0159.
 - [20] J. G. Baker *et al.*, *Phys. Rev. D* **78**, 044046 (2008).
 - [21] A. Buonanno, L. E. Kidder, and L. Lehner, *Phys. Rev. D* **77**, 026004 (2008).
 - [22] J. D. Schnittman, *Phys. Rev. D* **70**, 124020 (2004).
 - [23] We use a standard deviation of $\sigma = \pi/20$, but almost identical results are obtained for $\pi/40 \leq \sigma \leq \pi/5$.