Final Exam

1 a) For $|a|<1$, we write

$$f(\theta) = \ln\left(1 - ae^{i\theta} - ae^{-i\theta} + a^2\right)$$

$$= \ln\left((1-ae^{i\theta})(1-ae^{-i\theta})\right)$$

$$= \ln\left(1-ae^{i\theta}\right) + \ln\left(1-ae^{-i\theta}\right))$$

$$= -\sum_{n=1}^{\infty}\left(ae^{i\theta}\right)^n - \sum_{n=1}^{\infty}\left(ae^{-i\theta}\right)^n$$

$$= \sum_{n=1}^{\infty} -a^n\left(e^{in\theta} + e^{-in\theta}\right)$$

For $|a|>1$, we write

$$f(\theta) = \ln\left(a^2\left(a^{-2} - 2a^{-1} + 1\right)\right)$$

$$= 2\ln|a| + \sum_{n=1}^{\infty} -a^{-n}\left(e^{in\theta} + e^{-in\theta}\right)$$

b) The integral of $f(\theta)$ is $2\pi f_0$, the $n=0$ amplitude in the series. This vanishes for $|a|<1$, and equals $4\pi\ln|a|$ for $|a|>1$.

2 To calculate the principal value, we write

$$\int_{-\infty}^{\infty}\frac{x\sin x}{x^2-a^2}\,dx = \mathrm{Im}\int_{-\infty}^{\infty} e^{ix}\,\frac{1}{2}\left(\frac{1}{x-a} + \frac{1}{x+a}\right)dx$$

$$= \mathrm{Im}\left[i\pi\cdot\frac{1}{2}e^{ia} + i\pi\cdot\frac{1}{2}e^{-ia}\right]$$

$$= \pi\cos a$$

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This uses the half-residue formula. We can also move the poles off the real axis by adding a small imaginary part $i\varepsilon_\pm$, which may be positive or negative. Only in the former case will they contribute to the integral since we must close in the upper half-plane. We have

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - a^2}\, dx \rightsquigarrow \mathrm{Im}\left[\frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{1}{x-(a+i\varepsilon_+)} + \frac{1}{x-(-a+i\varepsilon_-)}\right) e^{ix}\,dx\right]$$

$$= \frac{1}{2}\,\mathrm{Im}\left[2\pi i\, e^{i(a+i\varepsilon_+)}\,\theta(\varepsilon_+) + 2\pi i\, e^{i(-a+i\varepsilon_-)}\,\theta(\varepsilon_-)\right]$$

$$\to \pi \cos a\,\left[\theta(\varepsilon_+) + \theta(\varepsilon_-)\right] \quad \text{as} \quad \varepsilon_\pm \to 0.$$

Thus, we get values $2\pi \cos a$, $\pi \cos a$ (two different ways), and 0. The average of these is indeed the principal value.

3 a) The generating function is

$$e^{(t-t^{-1})z/2} = \sum_{n=-\infty}^{\infty} J_n(z)\, t^n,$$

which is a Laurent series in t. The integral for extracting it coefficients gives

$$\oint_c t^{-(n+1)}\, e^{(t-t^{-1})z/2}\, dt = 2\pi i\, J_n(z)$$

because the only singularities are at the origin and infinity (both essential). The result follows.

b) The transformation $t \to -t^{-1}$ preserves the argument of the exponential, and we find

$$J_{-n}(z) = \frac{1}{2\pi i}\oint_c e^{(t-t^{-1})z/2}\, t^{n-1}\, dt$$

$$= \frac{-1}{2\pi i}\oint_c e^{(-t^{-1}+t)z/z}\,(-t^{-1})^{n-1}\, t^{-2}\, dt$$

$$= \frac{(-1)^n}{2\pi i}\oint_c e^{(t-t^{-1})z/2}\, t^{-(n+1)}\, dt$$

$$= (-1)^n J_n(z)$$

The sign outside the integral in the second line arises because $e^{i\theta} \to -e^{-i\theta}$ reverses the orientation of the integral.

c) When $t = e^{i\theta}$, $t - t^{-1} = e^{i\theta} - e^{-i\theta} = 2i\sin\theta$, which is pure imaginary. Therefore,

$$\int_0^\infty e^{-sz}\, e^{(t-t^{-1})z/2}\, dz = \frac{1}{s-(t-t^{-1})/2}$$

converges when $\mathrm{Re}[s] > 0$.

d) We can write

$$L[J_n](s) = \int_0^\infty e^{-sz}\,\frac{(-1)^n}{2\pi i}\oint_c e^{(t-t^{-1})z/2}\, t^{n-1}\, dt\, dz$$

$$= \frac{(-1)^n}{2\pi i}\oint_c \frac{t^{n-1}\, dt}{s-(t-t^{-1})/2} = \frac{(-1)^n}{\pi i}\oint_c \frac{-t^n\, dt}{t^2-2st-1}$$

The denominator has simple roots at $t = s \pm \sqrt{s^2+1}$. Only the minus sign gives a pole inside the circle. The result follows immediately.

4 a) Here, we can pick the source point as the origin of coordinates, and set

$$Y(\vec{r}) = \frac{1}{r} f(r)$$

$$\Rightarrow (\nabla^2 - \mu^2) Y = \left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \mu^2\right)\frac{f}{r}$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial r} r - \mu^2\right)\frac{f}{r}$$

$$\Rightarrow \frac{1}{r}\left(f'' - \mu^2 f\right) = 0 \quad \Rightarrow \quad f \propto e^{-\mu r}$$

The exponential decay is fixed by demanding regularity at infinity. Thus,

$$Y(\vec{r}) = \frac{c}{r} e^{-\mu r} \quad \text{for some } c.$$

We can use a Gaussian integral to find the δ-source at the origin and fix c:

$$\int_B (\nabla^2 - \mu^2) Y \, dB \cong \oint_S \vec{\nabla} Y \cdot \hat{r} \, dS \cong -4\pi c$$

We have kept only leading-order terms here in the integral over a ball of small radius r. The result follows.

b) The radial wave equation in this case is

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \frac{1}{r^2}\ell(\ell+1) - \mu^2\right) f_\ell(r) = 0,$$

which is a modified Bessel equation in $z = \mu r$. The general form of the Green function expansion follows immediately.

The only question remaining concerns the amplitudes in the expansion

$$Y_{\vec{a}}(\vec{r}) = \sum_{\ell m} A_{\ell m}\, i_\ell(\mu r_<)\, K_\ell(\mu r_>)\, \overline{Y_{\ell m}(\hat{a})}\, Y_{\ell m}(\hat{r})$$

As discussed in class, these amplitudes are fixed by the Wronskian

$$W(i_\ell, K_\ell; z) := i_\ell(z) K_\ell'(z) - i_\ell'(z) K_\ell(z)$$

The asymptotic expansions give

$$i_\ell(z) \sim \frac{e^z}{2z} \quad \text{and} \quad K_\ell(z) \sim \frac{e^{-z}}{z}$$

$$\Rightarrow i_\ell'(z) \sim \left(1 - \tfrac{1}{z}\right) i_\ell(z) \qquad K_\ell'(z) \sim \left(-1 - \tfrac{1}{z}\right) K_\ell(z)$$

$$\Rightarrow W(i_\ell, K_\ell; z) \sim \left[\left(-1 - \tfrac{1}{z}\right) - \left(1 - \tfrac{1}{z}\right)\right] i_\ell(z) K_\ell(z)$$

$$\sim -2\, \frac{1}{2z^2} = \frac{-1}{z^2}$$

Now, integrating the radial equation from just inside $r=a$ to just outside gives

$$a^2 \frac{\partial}{\partial r}\left[i_\ell(\mu a) K_\ell(\mu r) - i_\ell(\mu r) K_\ell(\mu a)\right] = 1$$

$$= \mu a^2 \left[i_\ell(\mu a) K_\ell'(\mu a) - i_\ell'(\mu a) K_\ell(\mu a)\right]$$

$$= \mu a^2 \cdot \frac{-1}{(\mu a)^2} = \frac{-1}{\mu} \qquad \Rightarrow \qquad A_{\ell m} = -\mu$$

This gives the result.