Problem Set VI

1 a) The general harmonic potential inside the sphere may be written in the form

$$\Phi(r,\theta,\phi) = \sum_{\ell m} \Phi_{\ell m}\, r^{\ell}\, Y_{\ell m}(\theta,\phi)$$

Since this problem is azimuthally symmetric, only the $m=0$ modes contribute. Matching the boundary conditions, we find

$$\Phi(a,\theta,\phi) = \sum_{\ell} \Phi_{\ell 0}\, a^{\ell}\, Y_{\ell 0}(\theta,\phi)$$

$$\Rightarrow \quad \Phi_{\ell 0} = a^{-\ell} \oint_s \bar{Y}_{\ell 0}(\theta,\phi)\, \Phi(a,\theta,\phi) \sin\theta\, d\theta\, d\phi$$

$$= 2\pi a^{-\ell} \sqrt{\frac{2\ell+1}{4\pi}} \int_{-1}^{1} P_\ell(z)\, \Phi_a(z)\, dz$$

Here we have set $z=\cos\theta$ in the integral and used the definition (8.341) of the spherical harmonics. Continuing, we find

$$\Phi_{\ell 0} = 2\pi V_0 a^{-\ell} \sqrt{\frac{2\ell+1}{4\pi}} \left[\int_{z_0}^{1} P_\ell(z)\, dz - \int_{-1}^{-z_0} P_\ell(z)\, dz \right]$$

$$= 2\pi V_0 a^{-\ell} \sqrt{\frac{2\ell+1}{4\pi}} \left[\Big(P_{\ell+1}(z) - P_{\ell-1}(z) \Big)_{z_0}^{1} - \Big(P_{\ell+1}(z) - P_{\ell-1}(z) \Big)_{-1}^{-z_0} \right] \frac{1}{2\ell+1}$$

$$= \frac{2\pi}{2\ell+1} V_0 a^{-\ell} \sqrt{\frac{2\ell+1}{4\pi}} \Big[-\big(P_{\ell+1}(z_0) - P_{\ell-1}(z_0)\big) - \big(P_{\ell+1}(-z_0) - P_{\ell-1}(-z_0)\big) \Big]$$

We have used (8.323) to do the integral here. Clearly, we must have ℓ odd here, so that $P_{\ell+1}$ and $P_{\ell-1}$ are even functions, in order to get a non-zero coefficient.

Accordingly, we find

$$\Phi_{\ell 0} = \sqrt{\frac{4\pi}{2\ell+1}}\; V_0 a^{-\ell} \left[P_{\ell-1}(z_0) - P_{\ell+1}(z_0) \right]$$

$$\Rightarrow \Phi(r,\theta,\phi) = V_0 \sum_{\substack{\ell \\ \text{odd}}} \left(\frac{r}{a}\right)^{\ell} \left[P_{\ell-1}(z_0) - P_{\ell+1}(z_0) \right] P_\ell(\cos\theta)$$

b) The external potential merely replaces $\left(\frac{r}{a}\right)^\ell$ in the previous expression with $\left(\frac{a}{r}\right)^{\ell+1}$. The dipole moment is the coefficient of $P_1(\cos\theta)/r^2$, which is

$$p = V_0 a^2 \left[P_0(z_0) - P_2(z_0) \right]$$

$$= V_0 a^2 \left[1 - \tfrac{1}{2}(3z_0^2 - 1) \right] = \tfrac{3}{2} V_0 a^2 \sin^2\theta_0$$

2 a) The normal modes solve the Helmholtz equation

$$(\nabla^2 + \omega^2)\psi = 0 \qquad \text{with} \qquad \frac{\partial \psi}{\partial n} = 0$$

In this case, we separate variables to find the modes

$$\psi_{\ell mn}(x,y,z) = \cos\left(\frac{\ell\pi}{a}x\right)\cos\left(\frac{m\pi}{b}y\right)\cos\left(\frac{n\pi}{c}z\right)$$

$$\omega_{\ell mn}^2 = \pi^2\left(\frac{\ell^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2}\right)$$

Any degeneracies here are coincidental since the box has no symmetries in general. In fact, no degeneracies can occur at all unless some ratio of the side lengths a, b, c is rational because ℓ, m, n are non-negative integers.

b) Here, we find

$$\psi_{lmn}(r,\theta,\phi) = j_l\left(z'_{ln}\frac{r}{R}\right) Y_{lm}(\theta,\phi)$$

with z'_{ln} the n^{th} zero of the derivative $j'_l(z)$ of the spherical Bessel function. The frequencies are

$$\omega^2_{lmn} = \frac{z'^2_{ln}}{R^2}$$

These are independent of m because of the spherical symmetry of the cavity

c) Here, we have

$$\psi_{lmn}(r,\theta,z) = J_{|m|}\left(z'_{|m|n}\frac{r}{R}\right) e^{im\phi} \cos\left(\frac{l\pi}{L} z\right)$$

with z'_{mn} the n^{th} zero of $J'_m(z)$. The frequencies in this case are

$$\omega^2_{lmn} = \frac{z'^2_{mn}}{R^2} + \frac{l^2\pi^2}{L^2}$$

with l a non-negative integer. The spectrum is degenerate because, except for $m=0$, $\pm m$ are distinct modes with the same frequency.

3 a) Here, we consider the spherical modes

$$\psi_{lmn}(r,\theta,\phi) = j_l\left(z_{ln}\frac{r}{R}\right) Y_{lm}(\theta,\phi)$$

of the three-dimensional Laplacian. Here, z_{ln} is the n^{th} zero of $j_l(z)$, as demanded by the boundary condition $\psi = 0$ at $r = R$. Solving the diffusion equation, the time dependence is

$$\frac{1}{\kappa}\frac{\partial}{\partial t}\psi_{\ell m n} = \Big[\frac{z_{\ell n}^2}{R^2} j_\ell''\big(z_{\ell n}\tfrac{r}{R}\big) + \frac{2}{r}\frac{z_{\ell n}}{R} j_\ell'\big(z_{\ell n}\tfrac{r}{R}\big)$$

$$- \frac{\ell(\ell+1)}{r^2} j_\ell\big(z_{\ell n}\tfrac{r}{R}\big) + \lambda j_\ell\big(z_{\ell n}\tfrac{r}{R}\big)\Big] Y_{\ell m}(\theta,\phi)$$

$$= \Big[-\frac{z_{\ell n}^2}{R^2} + \lambda\Big] j_\ell\big(z_{\ell n}\tfrac{r}{R}\big) Y_{\ell m}(\theta,\phi)$$

$$\Rightarrow \psi_{\ell m n}(t,r,\theta,\phi) = e^{-\kappa\left(\frac{z_{\ell n}^2}{R^2} - \lambda\right)t} j_\ell\big(z_{\ell n}\tfrac{r}{R}\big) Y_{\ell m}(\theta,\phi)$$

An explosion will occur when the coefficient of t in the exponential is positive:

$$0 < \kappa\Big(\lambda - \frac{z_{\ell n}^2}{R^2}\Big) \Rightarrow R^2 > \frac{z_{\ell n}^2}{\lambda} \Rightarrow R > \frac{z_{\ell n}}{\sqrt{\lambda}}$$

Now, the lowest-lying zero is $z_{01} = \pi$. This defines the critical radius

b) We can solve this problem the same way by just looking for spherical modes that vanish on the equatorial plane. These will have $\ell + m$ odd. Accordingly, the lowest-lying allowed zero in this case is $z_{11} \cong 4.4934$. This sets the critical radius for a hemisphere.

If R is the hemisphere critical radius, but we form a full sphere, then the $\ell=0$ $n=1$ mode has time dependence

$$e^{\kappa\lambda\left(1 - \frac{z_{01}^2}{z_{11}^2}\right)t} \Rightarrow \tau = \frac{z_{11}^2}{(z_{11}^2 - z_{01}^2)\kappa\lambda}$$

4 a) Here we Fourier transform in time to find

$$-\frac{i\omega}{\kappa}\hat{\varphi}(\omega,z) = \frac{\partial^2}{\partial z^2}\hat{\varphi}(\omega,z)$$

$$\Rightarrow \quad \hat{\varphi}(\omega,z) \propto e^{\pm\sqrt{-i\omega/\kappa}\,z}$$

Now, the square roots here are

$$\sqrt{-i\,\mathrm{sgn}(\omega)\,|\omega|/\kappa} = e^{-i\,\mathrm{sgn}(\omega)\frac{\pi}{4}}\sqrt{\frac{|\omega|}{\kappa}}$$

$$= \left(1 - i\,\mathrm{sgn}(\omega)\right)\sqrt{\frac{|\omega|}{2\kappa}}$$

Physisically, we must choose the solution that decays as we go to greater depth, so

$$\hat{\varphi}(\omega,z) = \hat{\varphi}(\omega,0)\,e^{-(1-i\,\mathrm{sgn}(\omega))\sqrt{|\omega|/2\kappa}\,z}$$

$$\Rightarrow \varphi(t,z) = \int_{-\infty}^{\infty}\frac{d\omega}{\sqrt{2\pi}}\,\hat{\varphi}(\omega,0)\,e^{-\sqrt{|\omega|/2\kappa}\,z}\,e^{-i(\omega t - \mathrm{sgn}(\omega)\sqrt{|\omega|/2\kappa}\,z)}$$

$$= \int_{-\infty}^{\infty}\frac{d\omega}{\sqrt{2\pi}}\,\hat{\varphi}(\omega,0)\,e^{-\sqrt{|\omega|/2\kappa}\,z}\,e^{-i\omega\left(t - \frac{z}{\sqrt{2\kappa|\omega|}}\right)}$$

We can read off the penetration depth and phase delay from here quite easily:

$$d = \sqrt{\frac{2\kappa}{\omega}} \quad \text{and} \quad \tau(z) = \frac{z}{\sqrt{2\kappa\omega}} = \frac{z}{\omega d}$$

b) If we let $z=d$, for example, the phase delay time is $\tau = 1/\omega = T/2\pi$. If $T = 1$ year, then τ is almost 2 months. The depth goes like $\sqrt{T}$, so for daily variations, $d = 3\,m/\sqrt{365}$, which is about 16 cm. The phase delay at that depth is about 4 hours.

c) To leading order, the effects just superpose. The equation is linear, so the answer is obvious. I'm not quite sure what the book is getting at here.