Problem Set III

1 We set $z = e^{i\theta}$ on the unit circle C, and

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \oint_C \frac{dz/iz}{a + b(z+z^{-1})/2} = \frac{2}{ib}\oint_C \frac{dz}{z^2 + 2\frac{a}{b}z + 1}$$

The integrand has roots at $z = z_\pm = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1}$. Only z_+ lies inside the contour so, factoring the denominator into $(z - z_+)(z - z_-)$, we find

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2}{ib} \cdot 2\pi i \frac{1}{z_+ - z_-} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

We can now use parametric differentiation to get the other integrals:

$$\int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2} = -\frac{\partial}{\partial a}\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi a}{(a^2 - b^2)^{3/2}}$$

$$\int_0^{2\pi} \frac{\cos\theta \, d\theta}{(a + b\cos\theta)^2} = -\frac{\partial}{\partial b}\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{-2\pi b}{(a^2 - b^2)^{3/2}}$$

2 The average power over a cycle is

$$\frac{d\bar{P}}{d\Omega} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} K \sin^2\theta \frac{\cos^2\omega t \, dt}{(1 + \beta\cos\theta\sin\omega t)^5}$$

$$= \frac{K\sin^2\theta}{2\pi}\int_0^{2\pi} \frac{\cos^2\tau \, d\tau}{(1 + \beta\cos\theta\sin\tau)^5}$$

$$= \frac{K\sin^2\theta}{2\pi}\left[\int_0^{2\pi} \frac{d\tau}{(1 + \beta\cos\theta\cos\tau)^5} - \int_0^{2\pi} \frac{\cos^2\tau \, d\tau}{(1 + \beta\cos\theta\cos\tau)^5}\right]$$

In the last line, we used $\cos^2\tau = 1 - \sin^2\tau$ and shifted τ by $\pi/2$ to get integrals of the form considered in the previous problem.

Now, using the results of the previous problem,

$$\frac{\partial^2}{\partial a^2}\int_0^{2\pi}\frac{d\tau}{a+b\cos\tau} = -2\cdot -1\cdot\int_0^{2\pi}\frac{d\tau}{(a+b\cos\tau)^3}$$

$$= -\frac{\partial}{\partial a}\frac{2\pi a}{(a^2-b^2)^{3/2}} = 3\frac{2\pi a^2}{(a^2-b^2)^{5/2}} - \frac{2\pi}{(a^2-b^2)^{3/2}} = \frac{2\pi(2a^2+b^2)}{(a^2-b^2)^{5/2}}$$

$$\Rightarrow 2\int_0^{2\pi}\frac{\cos^2\tau\, d\tau}{(a+b\cos\tau)^3} = \frac{\partial^2}{\partial b^2}\int\frac{d\tau}{a+b\cos\tau} = \frac{2\pi(a^2+2b^2)}{(a^2-b^2)^{5/2}}$$

$$\Rightarrow \int_0^{2\pi}\frac{\sin^2\tau\, d\tau}{(a+b\cos\tau)^3} = \frac{1}{2}\frac{2\pi(a^2-b^2)}{(a^2-b^2)^{5/2}} = \frac{\pi}{(a^2-b^2)^{3/2}}$$

$$\Rightarrow \frac{\partial^2}{\partial a^2}\int_0^{2\pi}\frac{\sin^2\tau\, d\tau}{(a+b\cos\tau)^3} = -3\cdot -4\int_0^{2\pi}\frac{\sin^2\tau\, d\tau}{(a+b\cos\tau)^5}$$

$$= -\frac{\partial}{\partial a}\left(3\frac{\pi a}{(a^2-b^2)^{5/2}}\right) = 3\cdot 5\frac{\pi a^2}{(a^2-b^2)^{7/2}} - 3\frac{\pi}{(a^2-b^2)^{5/2}}$$

$$= \frac{3\pi(4a^2+b^2)}{(a^2-b^2)^{7/2}} \Rightarrow \int_0^{2\pi}\frac{\sin^2\tau}{(a+b\cos\tau)^5} = \frac{\pi(4a^2+b^2)}{4(a^2-b^2)^{7/2}}$$

$$\Rightarrow \frac{d\bar{P}}{d\Omega} = \frac{K\sin^2\theta}{2\pi}\cdot\frac{\pi(4+\beta^2\cos^2\theta)}{4(1-\beta^2\cos^2\theta)^{7/2}}$$

3 a) Here, we can take

$$\int_0^\infty\frac{\cos ax\, dx}{(x^2+b^2)^2} = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos ax\, dx}{(x^2+b^2)^2}$$

$$= \frac{1}{2}\,\mathrm{Re}\left[\int_{-\infty}^{\infty}\frac{e^{iax}\,dx}{(x^2+b^2)^2}\right] = \frac{1}{2}\,\mathrm{Re}\left[\oint_C\frac{e^{iaz}\,dz}{(z^2+b^2)^2}\right]$$

where C is the closed great semi-circle in the upper half plane. The residue theorem then gives

$$\int_0^\infty\frac{\cos ax\, dx}{(x^2+b^2)^2} = \frac{1}{2}\,\mathrm{Re}\left[2\pi i\,\frac{d}{dz}\frac{e^{iaz}}{(z+ib)^2}\bigg|_{z=ib}\right]$$

$$= \pi\,\mathrm{Re}\left[\frac{-a e^{-ab}}{(2ib)^2} - 2\frac{ie^{-ab}}{(2ib)^3}\right] = \frac{\pi(ab+1)}{4b^3}e^{-ab}$$

b) Here, we cut out the removable singularity at $x=0$ to find

$$\int_0^\infty \frac{\sin ax \, dx}{x(x^2+b^2)} = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin ax \, dx}{x(x^2+b^2)} = \frac{1}{2}\,\mathrm{Im}\left[\oint \frac{e^{iax}\,dx}{x(x^2+b^2)}\right]$$

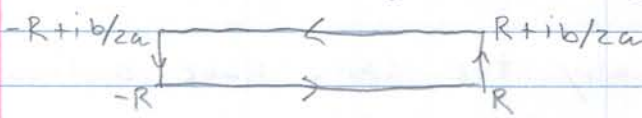
$$= \frac{1}{2}\,\mathrm{Im}\left[2\pi i\,\frac{e^{-ab}}{ib(2ib)} + \pi i\,\frac{1}{b^2}\right] = \frac{\pi}{2b^2}\left(1-e^{-ab}\right)$$

4 a) Here, we complete the square in the exponent:

$$\int_0^\infty e^{-ax^2}\cos bx \, dx = \frac{1}{2}\,\mathrm{Re}\left[\int_{-\infty}^\infty e^{-ax^2} e^{ibx}\,dx\right]$$

$$= \frac{1}{2}\,\mathrm{Re}\left[\int_{-\infty}^\infty e^{-a(x-ib/2a)^2 - b^2/4a}\,dx\right]$$

We need to translate the integrand along the imaginary axis to complete this problem. We do this by considering the contour integral over the large rectangle shown here. The



integrand has no singularities, so we need only show that the integrals over the vertical legs can be neglected:

$$\int_0^{b/2a} e^{-a(\pm R+iy-ib/2a)^2 - b^2/4a}\, i\,dy$$

$$= i e^{-aR^2 - b^2/4a}\int_0^{b/2a} e^{a(y-b/2a)^2 \mp 2iaR(y-b/2a)}\,dy$$

The remaining integral is bounded in modulus, so the whole thing is exponentially suppressed as $R\to\infty$. Thus, the integral along the lower edge of the rectangle is cancelled by that along the upper. Therefore, we write

$$\int_0^\infty e^{-ax^2} \cos bx \, dx = \frac{1}{2} \text{Re} \left[\int_{-\infty + ib/2a}^{\infty + ib/2a} e^{-a(x - ib/2a)^2 - b^2/4a} dx \right]$$

$$= \frac{1}{2} e^{-b^2/4a} \text{Re} \left[\int_{-\infty}^{\infty} e^{-ax^2} dx \right] = \frac{1}{2} \sqrt{\frac{\pi}{a}} e^{-b^2/4a}$$

b) First, we take $-\frac{\partial}{\partial b}$ of our result to find

$$\int_0^\infty e^{-ax^2} \cdot x \sin bx \, dx = \frac{\sqrt{\pi}\, b}{4a^{3/2}} e^{-b^2/4a}$$

We can now hit this with $\left(-\frac{\partial}{\partial a}\right)^n$ to bring down n powers of x^2 from the Gaussian. Thus, for example,

$$\int_0^\infty e^{-ax^2} \cdot x^3 \sin bx \, dx = \frac{\sqrt{\pi}\, b}{4} \left(\frac{3}{2} a^{-5/2} - a^{-3/2} \frac{b^2}{4a^2} \right) e^{-b^2/4a}$$

$$= \frac{\sqrt{\pi}\, b}{16} \frac{6a - b^2}{a^{7/2}} e^{-b^2/4a}$$

5 a) Here we use basically the same scheme from the example in class, which gave

$$\int_0^\infty \frac{\ln x}{a^2 + x^2} dx = \frac{\pi \ln a}{2a}$$

We take our branch line along the negative real axis and write

$$\int_{-\infty}^{\infty} \frac{(\ln x)^2}{a^2 + x^2} dx = \int_{-\infty}^{-\varepsilon} \frac{(\ln -x + i\pi)^2}{a^2 + x^2} dx + \int_{\varepsilon}^{\infty} \frac{(\ln x)^2}{a^2 + x^2} dx$$

$$= 2 \int_{\varepsilon}^{\infty} \frac{(\ln x)^2}{a^2 + x^2} dx + 2i\pi \int_{\varepsilon}^{\infty} \frac{\ln x}{a^2 + x^2} dx + \int_{\varepsilon}^{\infty} \frac{(i\pi)^2}{a^2 + x^2} dx$$

$$= 2 \int_0^\infty \frac{(\ln x)^2}{a^2 + x^2} + i \frac{\pi^2 \ln a}{a} - \frac{\pi^2}{2} \cdot \frac{2\pi i}{2ia}$$

Meanwhile, the residue theorem gives

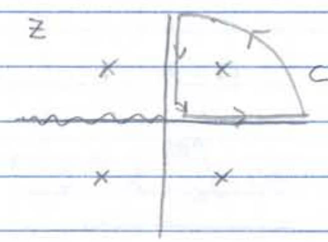
$$\int_{-\infty}^{\infty} \frac{(\ln x)^2}{a^2+x^2}\, dx = 2\pi i \frac{(\ln ia)^2}{2ia} = \frac{\pi(i\pi/2+\ln a)^2}{a}$$

$$\Rightarrow \int_0^{\infty} \frac{(\ln x)^2}{a^2+x^2} = \frac{\pi}{2a}\left((\ln a)^2 + i\pi \ln a - \frac{\pi^2}{4}\right) - i\frac{\pi^2 \ln a}{2a} + \frac{\pi^3}{4a}$$

$$= \frac{\pi(\ln a)^2}{2a} + \frac{\pi^3}{8a}$$

b) To avoid divergences near $x=0$, we must have $\mathrm{Re}[a]<1$. To avoid divergences as $x\to\infty$, we must have $\mathrm{Re}[a]>-3$. Thus, the convergence conditions are

$$-3 < \mathrm{Re}[a] < 1$$



The diagram at the right shows the branch cut, poles and integration contour we will use here. We have

$$\oint_C \frac{z^{-a}\,dz}{1+z^4} = \int_\varepsilon^R \frac{x^{-a}\,dx}{1+x^4} + \int_R^\varepsilon \frac{(iy)^{-a}\,d(iy)}{1+(iy)^4}$$

$$+ \int_0^{\pi/2} \frac{R^{-a}e^{-ia\theta}\,iRe^{i\theta}\,d\theta}{1+R^4e^{4i\theta}} \leftarrow \mathcal{O}(R^{-(a+3)})$$

$$+ \int_{\pi/2}^{0} \frac{\varepsilon^{-a}e^{-ia\theta}\,i\varepsilon e^{i\theta}\,d\theta}{1+\varepsilon^4 e^{4i\theta}} \leftarrow \mathcal{O}(\varepsilon^{1-a})$$

The two arc integrals vanish by our convergence conditions. Writing $i=e^{i\pi/2}$ and calculating the integral on the right using the residue at $z_0=e^{i\pi/4}$, we find

$$2\pi i \frac{z_0^{-a}}{(z_0-iz_0)(z_0+z_0)(z_0+iz_0)} = \left(1-ie^{-ia\pi/2}\right)\int_0^{\infty} \frac{x^{-a}\,dx}{1+x^4}$$

Thus, we find

$$\int_0^\infty \frac{x^{-a}\,dx}{1+x^4} = \frac{\pi}{2} i\, e^{-i(a+3)\pi/4}\left(1+e^{-i(a+1)\pi/2}\right)^{-1}$$

$$= \frac{\pi}{2} e^{-i(a+1)\pi/4}\left(1+e^{-i(a+1)\pi/2}\right)^{-1}$$

$$= \frac{\pi}{2}\left(e^{i(a+1)\pi/4}+e^{-i(a+1)\pi/4}\right)^{-1}$$

$$= \frac{\pi}{4} \sec\left((a+1)\frac{\pi}{4}\right)$$

6 a) Here, we do a variable substitution

$$u = x^n \Rightarrow x = u^{1/n} \Rightarrow dx = \frac{1}{n} u^{1/n - 1}\, du$$

$$\Rightarrow \int_0^\infty x^m e^{-x^n}\,dx = \int_0^\infty u^{m/n} e^{-u} \cdot \frac{1}{n} u^{1/n-1}\,du$$

$$= \frac{1}{n}\int_0^\infty e^{-u} u^{m+1/n - 1}\,du = \frac{1}{n}\Gamma\left(\frac{m+1}{n}\right)$$

b) Here, we take the result of (2.90)

$$\int_0^\infty \frac{w^\alpha\,dw}{e^{\beta w}-1} = \beta^{-(\alpha+1)}\,\Gamma(\alpha+1)\,\zeta(\alpha+1)$$

for $Re[\alpha] > 0$ and $Re[\beta] > 0$. Differentiating with respect to β gives

$$\int_0^\infty -\frac{w e^{\beta w} w^\alpha\,dw}{(e^{\beta w}-1)^2} = -(\alpha+1)\,\beta^{-(\alpha+2)}\,\Gamma(\alpha+1)\,\zeta(\alpha+1)$$

We now recall that $z\Gamma(z) = \Gamma(z+1)$ and shift $\alpha \to \alpha - 1$ to find

$$\int_0^\infty \frac{e^{\beta w} w^\alpha\,dw}{(e^{\beta w}-1)^2} = \beta^{-(\alpha+1)}\,\Gamma(\alpha+1)\,\zeta(\alpha)$$

7 To normalize the distribution, we solve

$$1 = N \int_0^1 x^{a-1}(1-x)^{b-1}\,dx = N\,B(a,b) \Rightarrow N = \frac{1}{B(a,b)}$$

The expectations are

$$\langle x \rangle = N \int_0^1 x^{a}(1-x)^{b-1}\,dx = N\,B(a+1,b) = \frac{B(a+1,b)}{B(a,b)}$$

$$\langle x^2 \rangle = N \int_0^1 x^{a+1}(1-x)^{b-1}\,dx = \frac{B(a+2,b)}{B(a,b)}$$

We can simplify these using $\Gamma(z+1) = z\,\Gamma(z)$ and $B(p,q) = \Gamma(p)\,\Gamma(q)/\Gamma(p+q)$:

$$B(p+1,q) = \frac{\Gamma(p+1)\,\Gamma(q)}{\Gamma(p+q+1)} = \frac{p\,\Gamma(p)\,\Gamma(q)}{(p+q)\,\Gamma(p+q)} = \frac{p}{p+q}\,B(p,q)$$

Therefore, we find

$$\langle x \rangle = \frac{a}{a+b} \quad \text{and} \quad \langle x^2 \rangle = \frac{(a+1)a}{(a+b+1)(a+b)}$$

$$\Rightarrow \sigma^2 = \frac{a(a+1)}{(a+b)(a+b+1)} - \frac{a^2}{(a+b)^2}$$

$$= \frac{a(a+1)(a+b) - a^2(a+b+1)}{(a+b)^2(a+b+1)}$$

$$= \frac{a^2(a+b) + a(a+b) - a^2(a+b) - a^2}{(a+b)^2(a+b+1)}$$

$$= \frac{ab}{(a+b)^2(a+b+1)}$$