Problem Set II

1 We have

$$\frac{\partial f}{\partial r}(z) = f'(z)\, e^{i\theta} \quad \text{and} \quad \frac{\partial f}{\partial \theta}(z) = f'(z)\, i r e^{i\theta}$$

$$\Rightarrow \frac{\partial f}{\partial \theta} = i r \frac{\partial f}{\partial r}$$

$$\Rightarrow \frac{\partial R}{\partial \theta} e^{i\Theta} + i R e^{i\Theta} \frac{\partial \Theta}{\partial \theta} = i r \left(\frac{\partial R}{\partial r} + i R \frac{\partial \Theta}{\partial r} \right) e^{i\Theta}$$

$$\Rightarrow \frac{\partial R}{\partial \theta} + i R \frac{\partial \Theta}{\partial \theta} = - r R \frac{\partial \Theta}{\partial r} + i r \frac{\partial R}{\partial r}$$

The Cauchy-Riemann conditions follow when we take real and imaginary parts.

2 In each case here, we write

$$\frac{1}{t^2 - z^2} = \frac{1}{2z}\left(\frac{1}{t-z} - \frac{1}{t+z} \right)$$

Thus, the integrands here have simple poles at $t = \pm z$ and

$$\frac{1}{2\pi i} \oint_C \frac{t}{t^2 - z^2} f(t)\, dt = \frac{1}{2z}\left(z f(z)\, \theta_c(z) - (-z) f(-z)\, \theta_c(-z) \right)$$

$$= \frac{1}{2}\left(f(z)\, \theta_c(z) + f(-z)\, \theta_c(z) \right)$$

$$\frac{1}{2\pi i} \oint_C \frac{t^2 + z^2}{t^2 - z^2} f(t)\, dt = \frac{1}{2z}\left(2z^2 f(z)\, \theta_c(z) - 2z^2 f(-z)\, \theta_c(-z) \right)$$

$$= z \left(f(z)\, \theta_c(z) - f(-z)\, \theta_c(-z) \right)$$

where $\theta_c(z) = 1$ if z is inside C and vanishes otherwise.

3 a) The second term vanishes when z is inside C_a since then a^2/z^* is outside. The first term gives the result by the Cauchy integral formula. We therefore find

$$f(re^{i\theta}) = \frac{1}{2\pi i}\int_0^{2\pi}\left(\frac{f(ae^{i\phi})}{ae^{i\phi}-re^{i\theta}} - \frac{f(ae^{i\phi})}{ae^{i\phi}-a^2e^{i\theta}/r}\right)iae^{i\phi}d\phi$$

$$= \frac{a}{2\pi}\int_0^{2\pi}\frac{(re^{i\theta}-a^2e^{i\theta}/r)\,e^{i\phi}f(ae^{i\phi})\,d\phi}{a^2e^{2i\phi}+a^2e^{2i\theta}-are^{i(\phi+\theta)}-a^3e^{i(\phi+\theta)}/r}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{(r^2-a^2)\,f(ae^{i\phi})\,d\phi}{ra(e^{i(\phi-\theta)}+e^{i(\theta-\phi)})-r^2-a^2}$$

$$= \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{f(ae^{i\phi})\,d\phi}{a^2+r^2-2ar\cos(\phi-\theta)}$$

b) The general formula follows immediately from part (a) when we take the real part:

$$\psi(r,\theta) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{\psi(a,\phi)\,d\phi}{a^2+r^2-2ar\cos(\phi-\theta)}$$

Setting $r=0$, we find

$$\psi(0,\theta) = \frac{a^2}{2\pi}\int_0^{2\pi}\frac{\psi(a,\phi)\,d\phi}{a^2} = \frac{1}{2\pi}\int_0^{2\pi}\psi(a,\phi)\,d\phi$$

c) Here, we find

$$\psi(r,\theta) = \frac{a^2-r^2}{2\pi}\,V\int_{\phi_1}^{\phi_2}\frac{d\phi}{a^2+r^2-2ar\cos(\phi-\theta)}$$

The integral here can be done in terms of elliptic functions.

4 a) First, we have

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{1-z} - \frac{1}{2-z} = \frac{1}{1-z} - \frac{1}{2}\frac{1}{1-z/2}$$

When $|z|<1$, we have

$$f(z) = \sum_{k=0}^{\infty} z^k - \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k = \sum_{k=0}^{\infty}\left(1-2^{-(k+1)}\right) z^k$$

When $1<|z|<2$, we must use another expansion for the first term:

$$f(z) = \frac{-1}{z}\left(\frac{1}{1-z^{-1}}\right) - \frac{1}{2}\frac{1}{1-z/2} = \frac{-1}{z}\sum_{k=0}^{\infty} z^{-k} - \frac{1}{2}\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k$$

$$= -\sum_{k=-\infty}^{-1} z^k - \sum_{k=0}^{\infty} 2^{-(k+1)} z^k$$

When $|z|>2$, we have

$$f(z) = \frac{-1}{z}\left(\frac{1}{1-z^{-1}}\right) - \frac{1}{2}\left(\frac{-2}{z}\right)\left(\frac{1}{1-2/z}\right)$$

$$= \frac{-1}{z}\sum_{k=0}^{\infty} z^{-k} + \frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{2}{z}\right)^k$$

$$= \sum_{k=0}^{\infty}\left(2^k-1\right) z^{-(k+1)} = \sum_{k=-\infty}^{-2}\left(2^{-k-1}-1\right) z^k$$

b) Here, we write

$$f(z) = \frac{2z}{z^2-1} = \frac{1}{z-1} + \frac{1}{z+1} = \frac{1}{1+(z-2)} + \frac{1}{3+(z-2)}$$

$$= \frac{1}{1+(z-2)} + \frac{1}{3}\frac{1}{1+\frac{z-2}{3}}$$

When $|z-2|<1$, we have

$$f(z) = \sum_{k=0}^{\infty}(-1)^k (z-2)^k + \frac{1}{3}\sum_{k=0}^{\infty}(-1)^k\left(\frac{z-2}{3}\right)^k = \sum_{k=0}^{\infty}(-1)^k\left(1-3^{-(k+1)}\right)(z-2)^k$$

When $1<|z-2|<3$,

$$f(z)=\frac{1}{z-2}\,\frac{1}{1+\frac{1}{z-2}}+\frac{1}{3}\,\frac{1}{1+\frac{z-2}{3}}$$

$$=\frac{1}{z-2}\sum_{K=0}^{\infty}(-1)^K\left(\frac{1}{z-2}\right)^K+\frac{1}{3}\sum_{K=0}^{\infty}(-1)^K\left(\frac{z-2}{3}\right)^K$$

$$=\sum_{K=-\infty}^{-1}(-1)^{K+1}(z-2)^K+\sum_{K=0}^{\infty}\frac{(-1)^K}{3^{K+1}}(z-2)^K$$

When $|z-2|>3$,

$$f(z)=\frac{1}{z-2}\,\frac{1}{1+\frac{1}{z-2}}+\frac{1}{z-2}\,\frac{1}{1+\frac{3}{z-2}}$$

$$=\frac{1}{z-2}\sum_{K=0}^{\infty}(-1)^K\left(\frac{1}{z-2}\right)^K+\frac{1}{z-2}\sum_{K=0}^{\infty}\left(\frac{3}{z-2}\right)^K(-1)^K$$

$$=\sum_{K=-\infty}^{-1}(-1)^{K+1}\left(1+3^{-(K+1)}\right)(z-2)^K$$

c) This one is complicated because of the branch structure of $f(z)$, which potentially could make $f(z)$ discontinuous. We will assume in each case that the branch has been chosen such that $f(z)$ is continuous. So, for example, when $|z|<1$,

$$f(z)=\frac{i}{\sqrt{1-z^2}}=i\sum_{K=0}^{\infty}\frac{(2K+1)!!}{2^K}\,z^{2K}$$

where $(2K+1)!!=1\cdot3\cdot5\cdot\cdots\cdot(2K+1)$.

Meanwhile, when $|z|>1$,

$$f(z)=\frac{1}{z\sqrt{1-z^{-2}}}=\frac{1}{z}\sum_{K=0}^{\infty}\frac{(2K+1)!!}{2^K}\,z^{-2K}$$

$$=\sum_{K=0}^{\infty}\frac{(2K+1)!!}{2^K}\,z^{-(2K+1)}$$

d) I should not have assigned this problem. But here is a solution:

$$f(z) = \sin\left(z + \tfrac{1}{z}\right) = \frac{1}{2i}\left(e^{i(z+1/z)} - e^{-i(z+1/z)}\right)$$

$$= \frac{1}{2i}\left(e^{iz - 1/iz} - e^{-iz - 1/-iz}\right)$$

We will encounter exponentials like these much later on in the course, as the generating function for the Bessel functions:

$$e^{\frac{x}{2}(t - 1/t)} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n \qquad (8.152)$$

Using this formula here,

$$f(z) = \frac{1}{2i}\left(\sum_{n=-\infty}^{\infty} J_n(2)\,(iz)^n - \sum_{n=-\infty}^{\infty} J_n(2)\,(-iz)^n\right)$$

$$= \sum_{n \text{ odd}} J_n(2)\, i^{n-1} z^n$$

That is, the Laurent coefficients are Bessel functions $J_n(x)$, evaluated at $x = 2$.

5 a) This function has simple poles at $z = -1$ and $z = \infty$. To see the latter, note that $z^2/1+z \sim z$ as $z \to \infty$. This is equivalent to taking $1/z \to 0$, as the problem suggests.

b) This function looks as though it has a pole at $z=0$, but it doesn't. If we Taylor expand the cosine, we see that $\frac{1}{2}z$ is the lowest-order term in its Laurent series. The

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Laurent series does have terms of arbitrarily high order in z, however, so there is an essential singularity at $z=\infty$.

c) Taylor expanding the exponential reveals terms of arbitrarily high order in z^{-1}, so there is an essential singularity at $z=0$. The function goes like z as $z\to\infty$, so there is a simple pole there.

d) This function obviously has simple poles at $z=\pm ia$, and contains terms of arbitrarily high order in z as $z\to\infty$. Therefore, there is also an essential singularity at $z=\infty$.

6 a) This function has obvious poles of order 1 at $z=-2i$ and order 2 at $z=0$. The residue of $f(z)$ at $z=z_0$ is defined to be the coefficient of $(z-z_0)^{-1}$ in the Laurent expansion at that point, so

$$\mathrm{Res}(f(z),\ z=-2i) = \frac{-2i+1}{(-2i)^2} = -\frac{1}{4}+\frac{1}{2}i$$

At $z=0$, we need to be a little more sly!

$$f(z)=\frac{z+1}{z^2(z+2i)} = \frac{1}{z^2}\frac{1}{2i}\frac{1+z}{1+z/2i} = \frac{1}{z^2}\frac{1}{2i}(1+z)\left(1-\frac{z}{2i}+\cdots\right)$$

$$= \frac{1}{2i}\frac{1}{z^2} + \frac{1}{2i}\left(1-\frac{1}{2i}\right)\frac{1}{z} + \cdots$$

$$\Rightarrow \mathrm{Res}(f(z),\ z=0) = \frac{1}{4}-\frac{1}{2}i$$

b) we need first of all to know where $\tanh z$ is singular:

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{z}-e^{-z}}{e^{z}+e^{-z}} = \infty$$

$$\Rightarrow \quad e^{z}+e^{-z}=0 \quad \Rightarrow \quad e^{2z}=-1$$

$$\Rightarrow \quad e^{z}=\pm i \quad \Rightarrow \quad z = \ln \pm i + 2\pi i n = \frac{4n\pm 1}{2}\pi i$$

That is, the poles are at odd multiples of $\frac{\pi}{2} i$. Expanding about such a point, we set

$$z = (4n\pm 1)\frac{\pi}{2} i + \varepsilon \quad \Rightarrow \quad e^{z} = e^{2\pi i \pm i\frac{\pi}{2}}\, e^{\varepsilon} = \pm i e^{\varepsilon}$$

$$\Rightarrow \tanh z = \frac{\pm i e^{\varepsilon} - (\mp i) e^{-\varepsilon}}{\pm i e^{\varepsilon} + (\mp i) e^{-\varepsilon}} = \frac{e^{\varepsilon}+e^{-\varepsilon}}{e^{\varepsilon}-e^{-\varepsilon}}$$

$$= \frac{2+\varepsilon^{2}+\cdots}{2\varepsilon + \frac{1}{3}\varepsilon^{3}+\cdots} = \frac{1}{\varepsilon}\,\frac{1+\frac{1}{2}\varepsilon^{2}+\cdots}{1+\frac{1}{6}\varepsilon^{2}+\cdots}$$

Thus, the singular points are simple poles with residue equal to one.

We cannot call the point at infinity an essential singularity in this case because it is not isolated. The singular points of $\tanh w^{-1}$ satisfy

$$w^{-1} = \frac{4n\pm 1}{2}\pi i \quad \Rightarrow \quad w = \frac{-2i}{(4n\pm 1)\pi} \to 0 \quad \text{as } n\to\infty.$$

c) Here, we have simple poles at $z=\pm i\pi$ and an essential singularity at $z=\infty$. We have

$$\operatorname{Res}(f(z), z=\pm i\pi) = \frac{e^{\pm i\pi}}{(\pm i\pi)-(\mp i\pi)} = \frac{\pm i}{2\pi}$$

There are several ways to evaluate the residue at infinity. Let's begin with the direct route:

$$f(z) = \frac{e^z}{z^2+\pi^2} = \frac{1}{z^2}\frac{e^z}{1+(\pi/z)^2}$$

$$= \frac{1}{z^2}\sum_{j=0}^{\infty}\left(-\frac{\pi^2}{z^2}\right)^j \sum_{k=0}^{\infty}\frac{1}{k!}z^k$$

$$= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^j}{k!}\pi^{2j} z^{k-(2j+2)}$$

We are interested in terms that go like z^{-1}, so we set $k = 2j+1$ and find

$$\mathrm{Res}(f(z), z=\infty) = \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}\pi^{2j}$$

$$= \frac{1}{\pi}\sin\pi = 0$$

We could also find this result by noting that the sum of all three residues must vanish, so

$$0 = \mathrm{Res}(f(z), z=\infty) + \frac{i}{2\pi} + \frac{-i}{2\pi}$$

d) Once again, I should not have assigned this problem. The solution involves unfamiliar objects, this time the Bernoulli numbers.

The function here has a pole of order $n+1$ at $z=0$ and simple poles at $z = 2\pi i m$ with m a (non-zero) integer. The residues at the simple poles are obviously
$\mathrm{Res}(f(z), z=2\pi i m) = (2\pi i m)^{-n}$.

At $z=0$, we can use (1,231) to write

$$a_{-1} = \frac{1}{(n+1-1)!} \frac{d^{n+1-1}}{dz^{n+1-1}} \left(z^{n+1} \frac{1}{z^n(e^z-1)} \right)_{z=0}$$

$$= \frac{1}{n!} \frac{d^n}{dz^n} \frac{z}{e^z-1} \Big|_{z=0}$$

In words, the residue of $f(z)$ at $z=0$ is the coefficient of z^n in the Taylor expansion of $z/(e^z-1)$ about $z=0$. This expansion defines the Bernoulli numbers B_K via

$$\frac{z}{e^z-1} = \sum_{K=0}^{\infty} \frac{B_K}{K!} z^K$$

We have, for instance, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = \frac{-1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$, ... There is no formula for B_K in closed form.

7 a) Strictly, a meromorphic function has only singularities that are not merely isolated, they must be poles. Thus, in a neighborhood of every singular point z_p of $f(z)$ within C, we can write

$$f(z) = (z-z_p)^{-n} g(z)$$

with $g(z)$ analytic $g(z_p) \neq 0$ and n the order of the pole at z_p. Similarly, if $f(z)$ has a zero of order n at z_0,

$$f(z) = (z-z_0)^n h(z)$$

with $h(z)$ analytic and $h(z_0) \neq 0$. The logarithmic derivative $\phi(z) = f'(z)/f(z)$ has singularities at the poles of $f(z)$, where the numerator is undefined, and at the zeroes of $f(z)$, where the denominator vanishes. The integral over C can be written as a sum of integrals around the individual singularities of $\phi(z)$, so we need only calculate these residues:

$$\oint_{C_{z_p}} \frac{f'(z)}{f(z)}\,dz = \oint_{C_{z_p}} \frac{[(z-z_p)^{-n} g(z)]'}{(z-z_p)^{-n} g(z)}\,dz$$

$$= \oint_{C_{z_p}} \frac{-n(z-z_p)^{-(n+1)} g(z) + (z-z_p)^{-n} g'(z)}{(z-z_p)^{-n} g(z)}\,dz$$

$$= \oint_{C_{z_p}} \left[\frac{-n}{z-z_p} + \frac{g'(z)}{g(z)}\right] dz = -n \cdot 2\pi i$$

The second integral vanishes because both numerator and denominator are free of poles and $g(z) \neq 0$ in a sufficiently small neighborhood of z_p. An identical calculation shows that an n^{th}-order zero of $f(z)$ contributes $+n \cdot 2\pi i$ to the integral, and the result follows immediately.

b) The indefinite integral $\int \frac{f'(z)}{f(z)}\,dz = \ln f(z)$ is elementary. Thus, the integral of $\phi(z)$ around a closed curve must give the change in the value of $\ln f(z)$ as we track it around the curve. If $f(z)$ is single-valued, $\ln f(z)$ can only change by a multiple of $2\pi i$. Since the imaginary part of $\ln f(z)$ is $\arg f(z)$, the result follows immediately.