Problem Set 1

1. Setting $z = e^{i \theta}$, we have

$$\sum_{k=0}^{\infty} e^{ik\theta} = \frac{1 - e^{i(1+1)\theta}}{1 - e^{i\theta}} = \frac{e^{-i\theta/2} - e^{i(1+1)\theta}}{e^{-i\theta/2} - e^{i\theta/2}}$$

$$= \frac{e^{-i\theta/2} - e^{i\theta/2}}{-2i \sin \theta/2}$$

The denominator is now pure imaginary, so taking the real part of both sides gives the imaginary part of the numerator

$$\sum_{k=0}^{n} \cos(k\theta) = \frac{-\sin \theta/2 - \sin(n+\frac{1}{2})\theta}{-2 \sin \theta/2} = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2 \sin \theta/2}$$

2. The definition of the cosine gives

$$\cos z = w \implies \frac{1}{2}(e^{iz} + e^{-iz}) = w$$

$$\implies e^{iz} - 2w + e^{-iz} = 0 \implies (e^{iz})^2 - 2we^{iz} + 1 = 0$$

$$\implies e^{iz} = \frac{1}{2}(2w \pm \sqrt{(2w)^2 - 4}) = \frac{w \pm \sqrt{w^2 - 1}}{}$$

$$\implies z = -i \ln \left( \frac{w \pm \sqrt{w^2 - 1}}{} \right)$$

Setting $w = \alpha > 1$ gives a positive real number in the argument of the $\ln$ for either $\pm$ sign. The logs of these have an unambiguous real value, but recall that the complex log is defined only up to an integer multiple of $2\pi i$. Therefore, we find

$$z = \pi i \pm i \ln \left( \alpha \pm \sqrt{\alpha^2 - 1} \right)$$
When $a$ is negative, $a < -1$, we must pull a sign out of the argument of $\ln$ to get a calculable real number. Since $\ln(-1) = i\pi$,

$$z = (2n+1)i\pi - i \ln \left(-a \pm \sqrt{a^2 - 1}\right)$$

When $w = ia$, we have

$$z = -i \ln \left(i a \pm \sqrt{-a^2 - 1}\right) = -i \ln \left(i a \pm i \sqrt{a^2 + 1}\right)$$

$$= -i \ln (\pm i) - i \ln (\sqrt{a^2 + 1} \pm a)$$

$$= \pm \frac{i\pi}{2} + 2\pi n - i \ln \left(\sqrt{a^2 + 1} \pm a\right)$$

Once again, the last term has an unambiguous real value. It is gratifying that $\cos^{-1}w$ has two values in each $2\pi$ interval, even when it is necessarily complex.

We have

$$f(z) = \sqrt{(x+iy)^2 - 1} = \sqrt{x^2 - y^2 - 1 + 2ixy}$$

$$= \sqrt{(x^2 - y^2 - 1)^2 + 4x^2 y^2} \left( i \pm \tan^{-1} \frac{2xy}{x^2 - y^2 - 1} \right)$$

$$\Rightarrow V(x,y) = \sqrt{(x^2 - y^2 - 1)^2 + 4x^2 y^2} \cos \left( i \pm \tan^{-1} \frac{2xy}{x^2 - y^2 - 1} \right)$$

$$V(x,y) = \sqrt{(x^2 - y^2 - 1)^2 + 4x^2 y^2} \sin \left( i \pm \tan^{-1} \frac{2xy}{x^2 - y^2 - 1} \right)$$

Recall the trig identities $\cos \frac{\theta}{2} = \sqrt{\frac{1}{2} \left(1 + \tan^2 \frac{\theta}{2}\right)^{1/2}}$. $\sin \frac{\theta}{2} = \sqrt{\frac{1}{2} \left(1 + \tan^2 \frac{\theta}{2}\right)^{1/2}}$. $\cos \frac{\theta}{2} = \sqrt{\frac{1}{2} \left(1 + \tan^2 \frac{\theta}{2}\right)^{1/2}}$. $\sin \frac{\theta}{2} = \sqrt{\frac{1}{2} \left(1 + \tan^2 \frac{\theta}{2}\right)^{1/2}}$.
Therefore, we have

\[ \nu(x, y) = \frac{4}{\sqrt{(x^2 - y^2 - 1)^2 + 4x^2y^2}} \sqrt{\frac{1}{2} \left( 1 \pm \frac{4x^2y^2}{(x^2 - y^2 - 1)^2} \right)} \]

\[ = \sqrt{\frac{1}{2} \left( (x^2 - y^2 - 1)^2 + 4x^2y^2 \right)} \pm \sqrt{1 - (x^2 - y^2 - 1)^2} \]

The + sign gives \( u \), the –, \( v \).

Meanwhile, we have

\[ \sqrt{z + 1} = \sqrt{x + 1 + iy} = \frac{4}{\sqrt{(x+1)^2 + y^2}} e^{i \tan^{-1} \frac{y}{x+1}} \]

\[ = \frac{4}{\sqrt{(x+1)^2 + y^2}} \left( \sqrt{ \frac{1}{2} \left( 1 + \frac{1}{\sqrt{(x+1)^2 + y^2}} \right) } + i \sqrt{ \frac{1}{2} \left( 1 - \frac{1}{\sqrt{(x+1)^2 + y^2}} \right) } \right) \]

\[ = \sqrt{\frac{1}{2} \left( (x+1)^2 + y^2 \right)} + i \sqrt{\frac{1}{2} \left( (x+1)^2 + y^2 \right) - 1} \]

We can take products of these factors to compute \( u(x, y) \) and \( v(x, y) \) for \( g(z) \).

Note that \( f(z) \) has a branch cut wherever \( z^2 - 1 \) is real and negative, which happens if

(a) \( z \) is real and \( |z| < 1 \), or (b) \( z \) is pure imaginary.

\( g(z) \) has a branch cut only in the first case.

Therefore, take \( a > 0 \) real and

\[ f(-ia) = \sqrt{ -a^2 - 1 } = i \sqrt{1 + a^2} \]

\[ g(-ia) = \sqrt{ -1 + i a } \sqrt{ 1 - i a } = \frac{4}{\sqrt{1 + a^2}} e^{i \frac{1}{2} (\tan^{-1} a - \pi)} \]

\[ = \frac{4}{\sqrt{1 + a^2}} e^{i \frac{1}{2} \tan^{-1} a} \]

\[ = \sqrt{1 + a^2} e^{-i \frac{\pi}{2}} = -i \sqrt{1 + a^2} \]
4. a) As \( a \) ranges over real values,

\[
1 + ia = \sqrt{1 + a^2} \ e^{i \tan^{-1} a},
\]

where \(-\frac{\pi}{2} < \tan^{-1} a < \frac{\pi}{2}\). However, the numerator has a more complicated argument.

In terms of the same branch of \( \tan^{-1} \),

\[
-1 + ia = \sqrt{1 + a^2} \ e^{i \left( \tan^{-1} (-a) + \pi \text{sgn}(a) \right)}
\]

\[
= \sqrt{1 + a^2} \ e^{-i \tan^{-1} a} \ e^{i \pi \text{sgn}(a)}
\]

where \( \text{sgn}(a) \) is the sign \((\pm)\) of \( a \). Thus,

\[
\eta(a) = \left( e^{-2\pi \tan^{-1} a} \ e^{i \pi \text{sgn}(a)} \right) b = e^{2\pi \tan^{-1} a - b \pi \text{sgn}(a)}
\]

b) The discontinuity is obviously at \( a = 0 \), where \( \eta \) jumps from \( e^{b\pi} \) for \( a < 0 \) to \( e^{-b\pi} \) for \( a > 0 \).

The magnitude of the discontinuity is

\[
\Delta \eta(0) = e^{-b\pi} - e^{b\pi} = -2 \sinh(b\pi)
\]

5. a) We write

\[
f(z) = u(x, y) + i \eta(x, y)
\]

\[
= w(z^*) = w(x - iy) = u(x, -y) + i \eta(x, -y)
\]

\[
\Rightarrow \frac{\partial}{\partial x} (x, y) = \frac{2}{i} \frac{\partial u}{\partial x} (x, -y) \quad \text{and} \quad \frac{\partial}{\partial y} (x, y) = -\frac{2}{i} \frac{\partial \eta}{\partial y} (x, -y)
\]
Thus, if \( \frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) \) by the Cauchy-Riemann conditions, then \( \frac{\partial \bar{z}}{\partial x}(x,y) \neq \frac{\partial \bar{z}}{\partial y}(x,y) \).

b) In this case, we have \( u(x,y) = u(x,-y) \) and \( v(x,y) = -v(x,-y) \). Again, \( u \) and \( v \) do not satisfy the Cauchy-Riemann conditions.

c) In this case, \( u(x,y) = u(x,-y) \) and \( v(x,y) = -v(x,-y) \). It follows that

\[
\frac{\partial u}{\partial x}(x,y) = \frac{\partial u}{\partial x}(x,-y) = \frac{\partial v}{\partial y}(x,-y) = \frac{\partial v}{\partial y}(x,y)
\]

\[
\frac{\partial u}{\partial y}(x,y) = -\frac{\partial u}{\partial y}(x,-y) = \frac{\partial v}{\partial x}(x,-y) = -\frac{\partial v}{\partial x}(x,y)
\]

So, the Cauchy-Riemann conditions do not hold.

f(\( \bar{z} \)) is analytic.

6 a) \( \nabla^2(x^2 - y^2) = 6x - 6y \neq 0 \), so there can be no analytic \( f(z) \) in this case.

b) By observation, we see that this \( u(x,y) \) is the real part of \( f(z) = z^2 - i\bar{z} \), which of course is analytic.