Problem Set I

1 Setting $z = e^{i\theta}$, we have

$$\sum_{k=0}^{n} e^{ik\theta} = \frac{1 - e^{i(n+1)\theta}}{1 - e^{i\theta}} = \frac{e^{-i\theta/2} - e^{i(n+1/2)\theta}}{e^{-i\theta/2} - e^{i\theta/2}}$$

$$= \frac{e^{-i\theta/2} - e^{i(n+1/2)\theta}}{-2i\sin\theta/2}$$

The denominator is now pure imaginary, so taking the real part of both sides gives the imaginary part of the numerator

$$\sum_{k=0}^{n} \cos(k\theta) = \frac{-\sin\frac{\theta}{2} - \sin(n+\frac{1}{2})\theta}{-2\sin\frac{\theta}{2}} = \frac{1}{2} + \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{1}{2}\theta}.$$

2 The definition of the cosine gives

$$\cos z = w \quad \Rightarrow \quad \tfrac{1}{2}(e^{iz} + e^{-iz}) = w$$

$$\Rightarrow e^{iz} - 2w + e^{-iz} = 0 \quad \Rightarrow \quad (e^{iz})^2 - 2we^{iz} + 1 = 0$$

$$\Rightarrow e^{iz} = \tfrac{1}{2}\left(2w \pm \sqrt{(2w)^2 - 4}\right) = w \pm \sqrt{w^2 - 1}$$

$$\Rightarrow z = -i\ln\left(w \pm \sqrt{w^2 - 1}\right)$$

Setting $w = a > 1$ gives a positive real number in the argument of the ln for either $\pm$ sign. The logs of these have an unambiguous real value, but recall that the complex log is defined only up to an integer multiple of $2\pi i$. Therefore, we find

$$z = 2n\pi - i\ln\left(a \pm \sqrt{a^2 - 1}\right)$$

2 When a is negative, $a<-1$, we must pull a sign out of the argument of ln to get a calculable real number, since $\ln(-1) = i\pi$,

$$z = (2n+1)\pi - i\ln\left(-a \pm \sqrt{a^2-1}\right)$$

When $w = ia$, we have

$$z = -i\ln\left(ia \pm \sqrt{-a^2-1}\right) = -i\ln\left(ia \pm i\sqrt{a^2+1}\right)$$

$$= -i\ln(\pm i) - i\ln\left(\sqrt{a^2+1} \pm a\right)$$

$$= \pm\frac{\pi}{2} + 2\pi n - i\ln\left(\sqrt{a^2+1} \pm a\right)$$

Once again, the last term has an unambiguous real value. It is gratifying that $\cos^{-1} w$ has two values in each 2π interval, even when it is necessarily complex.

3 We have

$$f(z) = \sqrt{(x+iy)^2-1} = \sqrt{x^2-y^2-1+2ixy}$$

$$= \sqrt[4]{(x^2-y^2-1)^2+4x^2y^2}\; e^{i\frac{1}{2}\tan^{-1}\frac{2xy}{x^2-y^2-1}}$$

$$\Rightarrow \quad u(x,y) = \sqrt[4]{(x^2-y^2-1)^2+4x^2y^2}\,\cos\left(\frac{1}{2}\tan^{-1}\frac{2xy}{x^2-y^2-1}\right)$$

$$v(x,y) = \sqrt[4]{(x^2-y^2-1)^2+4x^2y^2}\,\sin\left(\frac{1}{2}\tan^{-1}\frac{2xy}{x^2-y^2-1}\right)$$

Recall the trig identities $\begin{matrix}\cos\\ \sin\end{matrix}\frac{\theta}{2} = \sqrt{\frac{1}{2}\left(1 \pm [1+\tan^2\theta]^{-1/2}\right)}$

Therefore, we have

$$\overset{u}{v}(x,y) = \sqrt[4]{(x^2-y^2-1)^2+4x^2y^2}\,\sqrt{\tfrac{1}{2}\left(1 \pm \frac{1}{\sqrt{1+\frac{4x^2y^2}{(x^2-y^2-1)^2}}}\right)}$$

$$= \sqrt{\tfrac{1}{2}\left(\sqrt{(x^2-y^2-1)^2+4x^2y^2} \pm |x^2-y^2-1|\right)}$$

The + sign gives u, the −, v.

Meanwhile, we have

$$\sqrt{z\pm 1} = \sqrt{x\pm 1+iy} = \sqrt[4]{(x\pm1)^2+y^2}\; e^{i\frac{1}{2}\tan^{-1}\frac{y}{x\pm 1}}$$

$$= \sqrt[4]{(x\pm1)^2+y^2}\left(\sqrt{\tfrac{1}{2}\left(1+\frac{1}{\sqrt{1+y^2/(x\pm1)^2}}\right)} + i\sqrt{\tfrac{1}{2}\left(1-\frac{1}{\sqrt{1+y^2/(x\pm1)^2}}\right)}\right)$$

$$= \sqrt{\tfrac{1}{2}\left(\sqrt{(x\pm1)^2+y^2}+|x\pm1|\right)} + i\sqrt{\tfrac{1}{2}\left(\sqrt{(x\pm1)^2+y^2}-|x\pm1|\right)}$$

We can take products of these factors to compute $u(x,y)$ and $v(x,y)$ for $g(z)$.

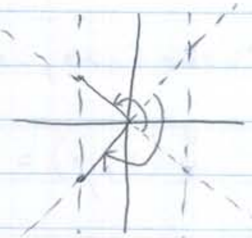
Note that $f(z)$ has a branch cut wherever z^2-1 is real and negative, which happens if (a) z is real and $|z|<1$, or (b) z is pure imaginary. $g(z)$ has a branch cut only in the first case. Therefore, take $a>0$ real and

$$f(\pm ia) = \sqrt{-a^2-1} = i\sqrt{1+a^2}$$

$$g(\pm ia) = \sqrt{-1\pm ia}\,\sqrt{1\pm ia} = \sqrt[4]{1+a^2}\, e^{i\frac{1}{2}(\tan^{-1}a-\pi)} \cdot \sqrt[4]{1+a^2}\, e^{-i\frac{1}{2}\tan^{-1}a}$$

$$= \sqrt{1+a^2}\, e^{-i\pi/2} = -i\sqrt{1+a^2}$$

4 a) As a ranges over real values,



$$1+ia = \sqrt{1+a^2}\, e^{i\tan^{-1} a},$$

where $-\frac{\pi}{2} < \tan^{-1} a < \frac{\pi}{2}$. However, the numerator has a more complicated argument. In terms of the same branch of $\tan^{-1}$,

$$-1+ia = \sqrt{1+a^2}\, e^{i(\tan^{-1}(-a) + \pi\,\mathrm{sgn}(a))}$$

$$= \sqrt{1+a^2}\, e^{-i\tan^{-1} a}\, e^{i\pi\,\mathrm{sgn}(a)},$$

where $\mathrm{sgn}(a)$ is the sign ($\pm$) of a. Thus,

$$\eta(a) = \left(e^{-2i\tan^{-1} a}\, e^{i\pi\,\mathrm{sgn}\, a}\right)^{ib} = e^{2b\tan^{-1} a - b\pi\,\mathrm{sgn}(a)}$$

b) The discontinuity is obviously at $a=0$, where η jumps from $e^{b\pi}$ for $a<0$ to $e^{-b\pi}$ for $a>0$. The magnitude of the discontinuity is

$$A\eta(0) = e^{-b\pi} - e^{b\pi} = -2\sinh(b\pi)$$

5 a) We write

$$f(z) = \xi(x,y) + i\eta(x,y)$$

$$= w(z^*) = w(x-iy) = u(x,-y) + i\,v(x,-y)$$

$$\Rightarrow\ \xi(x,y) = u(x,-y) \quad\text{and}\quad \eta(x,y) = v(x,-y)$$

$$\Rightarrow\ \frac{\partial\xi}{\partial x}(x,y) = \frac{\partial u}{\partial x}(x,-y) \quad\text{and}\quad \frac{\partial\eta}{\partial y}(x,y) = -\frac{\partial v}{\partial y}(x,-y)$$

Thus, if $\frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y)$ by the Cauchy-Riemann conditions, then $\frac{\partial \xi}{\partial x}(x,y) \neq \frac{\partial \eta}{\partial y}(x,y)$.

b) In this case, we have $\xi(x,y) = u(x,y)$ and $\eta(x,y) = -v(x,y)$. Again, ξ and η do not satisfy the Cauchy-Riemann conditions.

c) In this case, $\xi(x,y) = u(x,-y)$ and $\eta(x,y) = -v(x,-y)$
It follows that

$$\frac{\partial \xi}{\partial x}(x,y) = \frac{\partial u}{\partial x}(x,-y) = \frac{\partial v}{\partial y}(x,-y) = \frac{\partial \eta}{\partial y}(x,y)$$

$$\frac{\partial \xi}{\partial y}(x,y) = -\frac{\partial u}{\partial y}(x,-y) = \frac{\partial v}{\partial x}(x,-y) = -\frac{\partial \eta}{\partial x}(x,y)$$

So the Cauchy-Riemann conditions <u>do</u> hold. $f(\bar{z})$ is analytic.

6 a) $\nabla^2(x^3 - y^3) = 6x - 6y \neq 0$, so there can be no analytic $f(z)$ in this case.

b) By observation, we see that this $u(x,y)$ is the real part of $f(z) = z^2 - iz$, which of course is analytic.