Physics 5115 | Florida Atlantic University
Mathematical Physics | Fall, 2007

Problem Set II

Due: Monday, 17 September, 2007

1. Derive the Cauchy–Riemann equations in polar form. Let

$$z = r\,\mathrm{e}^{\mathrm{i}\theta} \qquad \text{and} \qquad f(z) = R(r,\theta)\,\mathrm{e}^{\mathrm{i}\Theta(r,\theta)}.$$

What equations relate the partial derivatives of R and Θ with respect to r and θ?

2. Suppose that $f(z)$ is analytic on and within the simple closed contour C. Evaluate

$$\frac{1}{2\pi\mathrm{i}} \oint_C \frac{t}{t^2 - z^2}\, f(t)\,\mathrm{d}t \qquad \text{and} \qquad \frac{1}{2\pi\mathrm{i}} \oint_C \frac{t^2 + z^2}{t^2 - z^2}\, f(t)\,\mathrm{d}t.$$

Consider the cases where z lies inside and outside the curve separately.

3. a. Suppose that $f(z)$ is analytic on and within the circle C_a of radius a about the origin. Prove that

$$f(z) = \frac{1}{2\pi i} \oint_{C_a} \left(\frac{f(s)}{s - z} - \frac{f(s)}{z - a^2/z^*} \right) \mathrm{d}s.$$

Deduce from this the *Poisson integral formula*

$$f(r,\theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(a,\phi)\,\mathrm{d}\phi}{a^2 + r^2 - 2ar\cos(\phi - \theta)}.$$

b. Suppose that we know the electrostatic potential $\psi(a,\phi)$ on the surface of a long, empty cylinder. (This will be a harmonic function in two dimensions; the longitudinal coordinate z drops out.) Obtain a general formula for the potential at any point within the cylinder. Show that the value of $\psi(0,\theta)$ at the center of the cylinder is equal to its average over the perimeter.

c. Compute the interior potential if $\psi(a,\phi)$ takes the constant value V for $\phi_0 < \phi < \phi_1$, and vanishes elsewhere on the cylinder.

4. For each of the following functions, construct a complete set of Laurent series about the specified point, and find the region of convergence for each series.

a. $f(z) = \dfrac{1}{(z-1)(z-2)}$ about $z = 0$,

b. $f(z) = \dfrac{2z}{z^2 - 1}$ about $z = 2$,

c. $f(z) = \dfrac{1}{\sqrt{z^2 - 1}}$ about $z = 0$, and

d. $f(z) = \sin(z + 1/z)$ about $z = 0$.

5. Classify the isolated singularities of

   a. $\dfrac{z^2}{1+z}$,

   b. $\dfrac{1-\cos z}{z}$,

   c. $z\,\mathrm{e}^{1/z}$, and

   d. $\dfrac{\mathrm{e}^{\mathrm{i}z}}{z^2+a^2}$.

   Consider the point at infinity in each case by taking $z = 1/w$ with $w \to 0$.

6. Locate the poles for

   a. $\dfrac{z+1}{z^2\,(z+2\mathrm{i})}$,

   b. $\tanh z$,

   c. $\dfrac{\mathrm{e}^z}{z^2+\pi^2}$, and

   d. $\dfrac{1}{z^n\,(\mathrm{e}^z-1)}$ (n an integer).

   Evaluate the residue at each pole.

7. Suppose that $f(z)$ is analytic and nonzero on the simple closed contour C and that it is meromorphic in the domain D within. Define the *logarithmic derivative* $\phi(z) := f'(z)/f(z)$ of $f(z)$ on D.

   a. Prove that
   $$\frac{1}{2\pi\mathrm{i}}\oint_C \phi(z)\,\mathrm{d}z = N_0 - N_p,$$
   where N_0 and N_p are the number of zeroes and poles, respectively, of $f(z)$ within D. Note that each accounts for multiplicity, so a double pole counts twice in N_p, for example.

   b. Show that
   $$\oint_C \phi(z)\,\mathrm{d}z = \mathrm{i}\,\Delta_C \arg f.$$
   That is, the integral above measures the change in the argument of the value of $f(z)$ as one makes a full loop over C.