Problem Set II
Due: Monday, 17 September, 2007

1. Derive the Cauchy–Riemann equations in polar form. Let

\[ z = r e^{i\theta} \quad \text{and} \quad f(z) = R(r, \theta) e^{i\Theta(r, \theta)}. \]

What equations relate the partial derivatives of \( R \) and \( \Theta \) with respect to \( r \) and \( \theta \)?

2. Suppose that \( f(z) \) is analytic on and within the simple closed contour \( C \). Evaluate

\[ \frac{1}{2\pi i} \oint_C \frac{t}{t^2 - z^2} f(t) \, dt \quad \text{and} \quad \frac{1}{2\pi i} \oint_C \frac{t^2 + z^2}{t^2 - z^2} f(t) \, dt. \]

Consider the cases where \( z \) lies inside and outside the curve separately.

3. a. Suppose that \( f(z) \) is analytic on and within the circle \( C_a \) of radius \( a \) about the origin. Prove that

\[ f(z) = \frac{1}{2\pi i} \oint_{C_a} \left( \frac{f(s)}{s - z} - \frac{f(s)}{z - a^2/z^*} \right) \, ds. \]

Deduce from this the Poisson integral formula

\[ f(r, \theta) = \frac{a^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(a, \phi) \, d\phi}{a^2 + r^2 - 2ar \cos(\phi - \theta)}. \]

b. Suppose that we know the electrostatic potential \( \psi(a, \phi) \) on the surface of a long, empty cylinder. (This will be a harmonic function in two dimensions; the longitudinal coordinate \( z \) drops out.) Obtain a general formula for the potential at any point within the cylinder. Show that the value of \( \psi(0, \theta) \) at the center of the cylinder is equal to its average over the perimeter.

c. Compute the interior potential if \( \psi(a, \phi) \) takes the constant value \( V \) for \( \phi_0 < \phi < \phi_1 \), and vanishes elsewhere on the cylinder.

4. For each of the following functions, construct a complete set of Laurent series about the specified point, and find the region of convergence for each series.

a. \( f(z) = \frac{1}{(z - 1)(z - 2)} \) about \( z = 0 \),

b. \( f(z) = \frac{2z}{z^2 - 1} \) about \( z = 2 \),

c. \( f(z) = \frac{1}{\sqrt{z^2 - 1}} \) about \( z = 0 \), and

d. \( f(z) = \sin(z + 1/z) \) about \( z = 0 \).
5. Classify the isolated singularities of
   a. \( \frac{z^2}{1 + z} \),
   b. \( \frac{1 - \cos z}{z} \),
   c. \( ze^{1/z} \), and
   d. \( \frac{e^z}{z^2 + a^2} \).

Consider the point at infinity in each case by taking \( z = \frac{1}{w} \) with \( w \to 0 \).

6. Locate the poles for
   a. \( \frac{z + 1}{z^2 (z + 2i)} \),
   b. \( \tanh z \),
   c. \( \frac{e^z}{z^2 + \pi^2} \), and
   d. \( \frac{1}{z^n (e^z - 1)} \) (n an integer).

Evaluate the residue at each pole.

7. Suppose that \( f(z) \) is analytic and nonzero on the simple closed contour \( C \) and that it is meromorphic in the domain \( D \) within. Define the logarithmic derivative \( \phi(z) := \frac{f'(z)}{f(z)} \) of \( f(z) \) on \( D \).
   a. Prove that
      \[
      \frac{1}{2\pi i} \oint_C \phi(z) \, dz = N_0 - N_p,
      \]
      where \( N_0 \) and \( N_p \) are the number of zeroes and poles, respectively, of \( f(z) \) within \( D \). Note that each accounts for multiplicity, so a double pole counts twice in \( N_p \), for example.
   b. Show that
      \[
      \oint_C \phi(z) \, dz = i \Delta_C \arg f.
      \]
      That is, the integral above measures the change in the argument of the value of \( f(z) \) as one makes a full loop over \( C \).