Physics 5115 | Florida Atlantic University
Mathematical Physics | Fall 2007

Final Exam

Due: Monday, December 10, 2007

- **Write your name on this cover page.**
- Exams are due by 6:00 PM on Monday.
- You may discuss these problems informally with other students in the class. However, working together is not allowed.
- Please contact me with any questions or concerns.
- Good luck!

1. For a given real constant $a \neq \pm 1$, define the function

$$f(\theta) := \ln(1 - 2a\cos\theta + a^2)$$

on the domain $-\pi < \theta < \pi$.

a. Calculate the Fourier coefficients f_n such that

$$f(\theta) = \sum_{n=-\infty}^{\infty} f_n \, e^{in\theta}.$$

Consider the cases $|a| > 1$ and $|a| < 1$ separately.
Hint: There is a *really* easy way to do this using complex variables. Rewrite the cosine and factor before using a well-known Taylor series.

b. Show that the integral of $f(\theta)$ over its entire domain vanishes if and only if $|a| < 1$. Calculate it for $|a| > 1$.

2. Find the principal value of

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - a^2} \, \mathrm{d}x.$$

Also calculate the values of this integral under the four possible deformations of the integration contour into the upper- or lower-half-plane at the poles $x = \pm a$ to avoid the singularities in the integrand. Check for consistency.

3. Calculate the Laplace transform of the Bessel function $J_n(x)$ with integer index n. You may want to proceed as follows.

   a. Recall the generating function for the Bessel functions and, for any integer n, use it to show that

   $$J_n(z) = \frac{1}{2\pi \mathrm{i}} \oint_C e^{(t-t^{-1})z/2}\, t^{-(n+1)}\, \mathrm{d}t,$$

   where C denotes the unit circle in the complex t-plane.

   b. Recover the identity $J_{-n}(z) = (-1)^n J_n(z)$ by changing the integration variable to $u := -t^{-1}$ in the integral above.
   Hint: What happens to the orientation of C under this change of integration variable? Beware the minus sign!

   c. Take $z = x$ to be real, and calculate the Laplace transform of the generating function. Argue that the transform integral converges for all real $s > 0$ when t lies on the unit circle.

   d. Finally, use the result of part (b) to express the Legendre transform of $J_n(x)$ in terms of a contour integral of your result from part (c). Find that

   $$L[J_n](s) = \frac{(\sqrt{1+s^2} - s)^n}{\sqrt{1+s^2}},$$

   with domain $s > 0$.

4. The static ($\omega = 0$) limit of the Klein–Gordon equation for a massive, relativistic particle in three dimensions is the modified Helmholtz equation

$$\boldsymbol{\nabla}^2\psi - \mu^2\,\psi = 0,$$

shown here in its homogeneous form.

a. Show that the Yukawa potential,

$$Y_{\mathbf{a}}(\mathbf{r}) = \frac{-e^{-\mu\|\mathbf{r}-\mathbf{a}\|}}{4\pi\,\|\mathbf{r}-\mathbf{a}\|},$$

is a Green function for this equation. That is, show that it has the point source $\delta_{\mathbf{a}}(\mathbf{r})$. What "boundary conditions" does $Y_{\mathbf{a}}(\mathbf{r})$ obey?

b. Show that the Yukawa potential can be expanded as

$$Y_{\mathbf{a}}(\mathbf{r}) = \sum_{\ell m} -\mu\, i_\ell(\mu r_<)\, k_\ell(\mu r_>)\, \overline{Y_{\ell m}(\hat{\mathbf{a}})}\, Y_{\ell m}(\hat{\mathbf{r}}),$$

where i_ℓ and k_ℓ are the modified spherical Bessel functions of the first and second kind, respectively.