Homework II

1 Let $L$ and $N$ clearly span the plane orthogonal to that spanned by $M$ and $M$. Thus, we only need to check

\[
L \cdot L = \frac{1}{2} (T \cdot T + 2 T \cdot Z + Z \cdot Z) = \frac{1}{2} (-1 + 0 + 1) = 0
\]
\[
N \cdot N = \frac{1}{2} (-1 + 1) = 0
\]
\[
L \cdot N = \frac{1}{2} (-1 - 1) = -1
\]
\[
M \cdot M = \frac{1}{2} (X \cdot X + 2iX \cdot Y - Y \cdot Y) = \frac{1}{2} (1 + 0 - 1) = 0
\]
\[
M \cdot M = \frac{1}{2} (1 - 1) = 0
\]
\[
M \cdot M = \frac{1}{2} (1 - i^2) = 1
\]

2 Let's solve problem 7.14 from d'Inverno's book first. We have

\[
\nabla_a \nabla_b X_c = \nabla_b \nabla_a X_c + R_{abc}^d X_d
\]
\[
= - \nabla_b \nabla_c X_a + R_{abc}^d X_d
\]

We have used the definition of the curvature in the first line, and Killing's equation $\nabla_a (a X_b) = 0$ in the second. The result has a cyclic permutation of indices, so we find

\[
\nabla_a \nabla_b X_c = - \left( - \nabla_c \nabla_a X_b + R_{bca}^d X_d \right) + R_{abc}^d X_d
\]
\[
= \nabla_c \nabla_a X_b + (R_{abc}^d - R_{bca}^d) X_d
\]
\[
= - \nabla_a \nabla_b X_c + (R_{abc}^d - R_{bca}^d + R_{ca}^d) X_d
\]
Using the first Bianchi identity \( R_{[abc]}^d = 0 \) and collecting the derivatives on the left gives the final result

\[
\nabla_a \nabla_b X_c = R_{c[b}^{\phantom{c[b}a} d X_d
\]

when we divide by 2 and interchange the first two indices on the curvature.

When spacetime is flat, the right side here vanishes, so

\[
\nabla_a \nabla_b X_c = 0 \Rightarrow \nabla_b X_c = w_{bac} = \text{const.}
\]

Since \( \nabla_c (b X_c) = 0 \) by Killing's equation, \( w_{bac} \) must be anti-symmetric. Integrating again gives the result

\[
X_c = x^b w_{bca} + t_c.
\]

In \( n \) dimensions, \( w_{bac} \) has \( \frac{1}{2} n(n-1) \) independent components, while \( t_c \) has \( n \). Therefore, a Killing vector is specified by \( \frac{1}{2} n(n+1) \) constants. In four dimensions, these ten constants describe four translations, three rotations and three boosts.

3. The unit basis vectors in the rotating frame rotate with the frame, so, for example

\[
\frac{d}{dt} \hat{v}' = \hat{w} \times \hat{v}'
\]
In the rotating frame, however, we take these to be constant. Thus,

\[ \frac{d}{dt} \hat{s}' = \frac{du_1}{dt} \hat{i}' + \frac{du_2}{dt} \hat{j}' + \frac{du_3}{dt} \hat{k}' \]

The result follows immediately by the Leibniz rule for the products \( u_1, \hat{i}' \), etc.

4 Let's take a different approach here. Using components in both frames, we write

\[ r^\alpha \hat{e}_\alpha = s^\alpha \hat{e}_\alpha + r'^\alpha \hat{e}'_\alpha \]

Taking two derivatives, and recalling that \( \hat{e}_\alpha = \hat{e}'_\alpha = \hat{w} \times \hat{e}_\alpha \), we find

\[ \ddot{r}^\alpha \hat{e}_\alpha = \ddot{s}^\alpha \hat{e}_\alpha + \dddot{r}^\alpha \hat{e}'_\alpha + 2 \dot{r}'^\alpha \hat{e}_\alpha + r'^\alpha \hat{e}_\alpha \]

\[ = \ddot{s}^\alpha \hat{e}_\alpha + \dddot{r}^\alpha \hat{e}_\alpha + 2 \dot{r}'^\alpha \hat{w} \times \hat{e}_\alpha \]

\[ + r'^\alpha \frac{d}{dt} (\hat{w} \times \hat{e}_\alpha) \]

The result follows when we set \( \ddot{s}' = r'^\alpha \hat{e}_\alpha \), \( \dddot{r}' = r'^\alpha \hat{e}_\alpha \) and \( \dddot{r}' = r'^\alpha \hat{e}_\alpha \), and recall that \( \ddot{r}' = \frac{r'^\alpha}{m} \) in the inertial frame.

5 The simplest generalization uses the torsion-free connection \( \nabla \epsilon \):

\[ \nabla_{\epsilon \epsilon} F_{\alpha \beta \gamma} = 0 \]
This is because the inertial coordinate connection $\partial a$ in Minkowski spacetime is the symmetric metric connection.

Now, using the Christoffel tensor $\Gamma^c_{ab}$ for $\nabla_a \partial a$, where $\partial a$ is any coordinate derivative on curved spacetime, we find

$$\nabla_a F_{bc} = \partial a F_{bc} + \Gamma^m_{ab} F_{mc} + \Gamma^m_{ac} F_{bm}$$

$$= \partial a F_{bc} - 2 \Gamma^m_{abc} F_{cm}$$

We have used anti-symmetry of $F_{ab}$ in the second line. When we anti-symmetrize over $a$ as well, however, this second term vanishes because $\nabla_a$ is torsion-free and thus $\Gamma^c_{ab} = \Gamma^c_{ba}$. Therefore,

$$\partial [a F_{bc}] = \nabla [a F_{bc}] = 0$$

Thus, the coordinate curl of $F_{bc}$ vanishes in all charts in all spacetimes.