Problem Set V
Due: Friday, 7 December 2007
(A date which will live in infamy!)

1. Recall that the extrinsic curvature of a two-dimensional submanifold \( S \) of a three-
   dimensional Riemannian manifold \( (\Sigma, q_{ab}) \) was defined in class to be
   \[
   \kappa^{ab} := -\sigma^{am} \nabla_m \hat{r}^b,
   \]
   where \( \hat{r}^b \) is the normal to \( S \) within \( \Sigma \) and \( \sigma^{am} \) is the two-dimensional metric on \( S \). Let \( \Sigma \) be ordinary Euclidean space, and \( S \) a round sphere of radius \( a \). Show that
   \[
   \kappa^{ab} = -a^{-1} \sigma^{ab}.
   \]
   Hint: Evaluate \( \nabla_a \nabla_b r^2 \) in (a) spherical and (b) Cartesian coordinates.

2. Consider the linearized plane gravitational wave \( h_{ab}(x) = A_{ab} e^{ik \cdot x} \) on Minkowski space-
time. Assume that it is in de Donder gauge, so that \( k^a A_{ab} = 0 \), but not necessarily in
   a transverse-traceless gauge.
   a. Let \( \hat{u}^a \) be a given time-like unit vector, and define the spatial unit vector \( \hat{k}^a \) by
      \[
      \hat{k}^a = \omega (\hat{u}^a + \hat{k}^a) \quad \text{and} \quad \hat{u}^a \hat{k}_a = 0.
      \]
      In words, \( \hat{k}^a \) is the unit vector along the direction of propagation for \( h_{ab}(x) \) within the spatial slice orthogonal to \( \hat{u}^a \). Define the projection operator
      \[
      P_{ab} := \eta_{ab} + \hat{u}_a \hat{u}_b - \hat{k}_a \hat{k}_b
      \]
      into the 2-plane orthogonal to both \( \hat{u}^a \) and \( \hat{k}^a \). Show that
      \[
      h_{ab}^{TT}(x) := A_{ab}^{TT} e^{ik \cdot x} \quad \text{with} \quad A_{ab}^{TT} := \left( P_a^m P_b^n - \frac{1}{2} P_{ab} P^{mn} \right) A_{mn}
      \]
      is a plane-wave solution of the homogeneous wave equation of linearized gravity, and that it obeys the complete set of transverse-traceless gauge conditions in the frame defined by \( \hat{u}^a \), including the de Donder condition, as defined in class. As the notation suggests, this operation projects out the transverse-traceless part of a plane wave initially given in an arbitrary (but de Donder) gauge.
   b. Show that the transverse-traceless part of the wave amplitude may also be calculated from the formula
      \[
      A_{ab}^{TT} = \left[ \bar{\eta}_{\hat{i}\hat{j}} - 2 \hat{k}^{(i} \bar{\eta}_{(a} \hat{k}_{b)} + \frac{1}{2} (\bar{\eta}_{ab} + \hat{k}_a \hat{k}_b) \hat{k}^i \hat{k}^j \right] \bar{\eta}_i \bar{\eta}_j - \frac{1}{3} \bar{\eta}_{ij} \bar{\eta}^{mn} A_{mn},
      \]
      where \( \bar{\eta}_{ab} := \eta_{ab} + \hat{u}_a \hat{u}_b \) denotes the spatial metric. Note that the first operator to act on \( A_{mn} \) yields the traceless part of its spatial projection.
3. Define the right- and left-handed circular polarization tensors by

\[ e_{\varnothing}^{ab} := \frac{1}{\sqrt{2}} (e_{ab}^+ + i e_{ab}^x) \quad \text{and} \quad e_{\varnothing}^{\times} := \frac{1}{\sqrt{2}} (e_{ab}^+ - i e_{ab}^x), \]

respectively. Consider a right-circularly-polarized gravitational plane wave with amplitude \( A \) propagating along the +z-direction. It is incident on a pair of non-accelerating particles separated by a distance \( d \) along the \( x \)-axis in the Minkowski background. Show that each particle moves in a right-handed circle relative to the other, as seen from above. What is the radius of that circle? What changes for a left-circularly-polarized wave? 

*Hint:* Recall that one must take the real part of the complex expression for the wave.

4. Consider a pair of identical point particles of mass \( m \) orbiting one another with frequency \( \omega \) in a circle of radius \( a \) about the origin in the \( xy \)-plane.

   a. Write the energy density of the source in terms of delta functions and show that the resulting trace-reversed metric perturbation field is

   \[ h_{ab}(t, \vec{r}) = \frac{8m}{r} u_a u_b - \frac{8ma^2\omega^2}{r} \left[ \cos(2\omega(t - r)) e_{ab}^{+z} + \sin(2\omega(t - r)) e_{ab}^{\times z} \right]. \]

   Here, \( e_{ab}^{+z} \) and \( e_{ab}^{\times z} \) are the “plus” and “cross” polarization tensors defined in class for waves propagating in the +z-direction.

   b. At large radius, the spherical waves produced by the source are very nearly planar, and we can extract their transverse traceless parts by assuming that the direction \( \hat{k}^a \) of propagation in the problems above is identical to the direction \( \hat{r}^a \) from the origin to the observation point. Show that an observer on the +z-axis will observe the wave

   \[ h_{ab}^{TT}(t, r\hat{z}) = -\frac{8\sqrt{2}ma^2\omega^2}{r} \Re \left[ e^{-2i\omega(t-r)} e_{ab}^{\times z} \right]. \]

   That is, an observer above the source will see right-circularly polarized waves.

   c. In contrast, show that an observer at large distance along the +x-axis will measure linearly polarized waves with

   \[ h_{ab}^{TT}(t, r\hat{x}) = \frac{4ma^2\omega^2}{r} \Re \left[ e^{-2i\omega(t-r)} e_{ab}^{+x} \right], \]

   where \( e_{ab}^{+x} := \hat{e}_a^y \hat{e}_b^y - \hat{e}_a^z \hat{e}_b^z \) defines the “plus” polarization along the +x-axis.