1. If we neglect material stresses, the stress-energy tensor of a slowly-moving matter distribution can be written, correct to first order in velocity, as

\[ T_{ab} = 2 \dot{t}_a (p_b) + (\dot{t}^c p_c) \dot{t}_a \dot{t}_b, \]

where \( p_a := -T_{ab} \dot{t}^b \) is the mass-energy current density four-vector and \( \dot{t}^a \) is the 4-velocity of a “Newtonian” observer. Let \( h_{ab} \) denote the solution of the post-Minkowski field equation in the de Donder gauge for this source. Show that

\[ A_a := -\frac{1}{4} h_{ab} \dot{t}^b \]

satisfies the Maxwell equations in Lorentz gauge with source \( J_a := p_a \). Furthermore, neglecting time-derivatives of the fields, show that

\[ h_{ab} = 4 \left[ 2 \dot{t}_a A_b + (\dot{t}^c A_c) \dot{t}_a \dot{t}_b \right] \]

for such a source.

2. Show that the geodesic equation applied to a perturbed metric of the form found in the previous problem yields an acceleration

\[ a = -E - 4v \times B, \]

correct to first order in the velocity of a test mass. Here, \( E \) and \( B \) are respectively the electric and magnetic fields on the “Newtonian” slices of spacetime derived from the 4-vector potential \( A_a \) using the standard formulae from electromagnetism.

3. A uniform, rigid, thin shell of radius \( R \) and mass \( M \ll R \) rotates slowly with angular velocity \( \omega \). Show that the electric and magnetic fields of the previous problem are

\[ E = 0 \quad \text{and} \quad B = \frac{2M}{3R} \omega \]

within the shell. An observer at rest of the center of this shell parallel propagates a spatial vector \( s^a \) with \( s^a \dot{t}_a = 0 \) along her world-line. Show that the inertial components of \( s^a \) precess according to

\[ \frac{ds}{dt} = \Omega \times s \quad \text{with} \quad \Omega = 2B = \frac{4M}{3R} \omega, \]

relative to transport in exact Minkoski spacetime. This effect roughly demonstrates the dragging of inertial frames by rotating bodies in general relativity.
4. The geometry outside a large \((R \gg M)\), static, spherical source may be written
\[
ds^2 = -(1 - \frac{2M}{r}) \, dt^2 + (1 + \frac{2M}{r}) \left( dx^2 + dy^2 + dz^2 \right),
\]
correct to first order, where \((t, x, y, z)\) are the inertial coordinates on the Minkowski background of the inertial observer who sees the source at rest.

a. Let \(k^a\) be the tangent to a null geodesic in an affine parameterization, and let \(t^a := \partial_t^a\). Show that \(e := -t^a k_a\) is constant along the geodesic.

b. An atom at rest on the surface of the sun emits a photon of frequency \(\omega_e\), which is absorbed by an atom at rest far from the sun. Show that the absorbed photon has a frequency \(\omega_r\) that is red-shifted by the amount
\[
z := \frac{\omega_e - \omega_r}{\omega_r} = \frac{M_{\odot}}{R_{\odot}} \approx 2 \times 10^{-6}.
\]
**Hint:** What are the four-velocities \(u_e^a\) and \(u_r^a\) of the sending a receiving atoms in their proper-time parameterizations?

5. Consider linearized gravity over a *curved* background geometry \(\tilde{g}_{ab}\). Show that, under a gauge transformation
\[
h_{ab} \mapsto \tilde{h}_{ab} := h_{ab} + 2 \tilde{\nabla}_a (\phi_b) - \tilde{g}_{ab} \tilde{\nabla}^c \phi_c,
\]
the connection perturbation transforms according to
\[
\tilde{\nabla}^a_{\ b} \mapsto \tilde{\tilde{\nabla}}^a_{\ b} := \tilde{\nabla}^a_{\ b} + \phi^m \tilde{R}^a_{\ mbc} - \tilde{\nabla}_a \tilde{\nabla}_b \phi^c.
\]
**Hint:** Recall the slightly simpler transformation law for the ordinary metric perturbation \(\tilde{g}_{ab}\), and note that \(\tilde{\nabla}^a_{\ b}\) is linear in \(\tilde{g}_{ab}\). You will need to use a Bianchi identity.

6. Consider, as in the previous problem, the action of a gauge transformation in linearized gravity over a curved background.

a. Using the result of the previous problem, show explicitly that
\[
\tilde{R}^{abc} \mapsto \tilde{\tilde{R}}^{abc} = \tilde{R}^{abc} + \mathcal{L}_\phi \tilde{R}^{abc}.
\]

b. Show explicitly, as a result of the previous part, that the Ricci and Einstein tensors transform analogously:
\[
\tilde{R}_{ab} \mapsto \tilde{\tilde{R}}_{ab} := \tilde{R}_{ab} + \mathcal{L}_\phi \tilde{R}_{ab} \quad \text{and} \quad \tilde{G}_{ab} \mapsto \tilde{\tilde{G}}_{ab} := \tilde{G}_{ab} + \mathcal{L}_\phi \tilde{G}_{ab}.
\]

c. Give general arguments, based on the tensorial character of \(G_{ab}(\lambda)\) and the action of a smooth family \(\Phi(\lambda)\) of diffeomorphisms on the spacetime geometry at each \(\lambda\), to support the latter transformation law above.

d. Use the general arguments of the previous part to show that the source tensor \(\tilde{T}_{ab}\) must transform according to
\[
\tilde{T}_{ab} \mapsto \tilde{\tilde{T}}_{ab} := \tilde{T}_{ab} + \mathcal{L}_\phi \tilde{T}_{ab}
\]
under a gauge transformation. Thus, show that the the first-order field equation is gauge-covariant even on a non-vacuum background.