Geometry Exercises II

1. a) For functions, we have

\[ L \bar{v} L \bar{w} f = L \bar{v} (\bar{w}(\bar{f})) = \bar{v}(\bar{w}(\bar{f})) \]

\[ \Rightarrow [L \bar{v}, L \bar{w}] (\bar{f}) = (\bar{v} \bar{w} - \bar{w} \bar{v}) (\bar{f}) = [\bar{v}, \bar{w}] (\bar{f}) \]

\[ = L [\bar{v}, \bar{w}] \bar{f} \]

For vector fields,

\[ L \bar{v} L \bar{w} \bar{X} = L \bar{v} [\bar{w}, \bar{X}] = [\bar{v}, [\bar{w}, \bar{X}]] \]

\[ \Rightarrow [L \bar{v}, L \bar{w}] \bar{X} = [\bar{v}, [\bar{w}, \bar{X}]] - [\bar{w}, [\bar{v}, \bar{X}]] \]

\[ = [\bar{v}, [\bar{w}, \bar{X}]] + [\bar{w}, [\bar{v}, \bar{X}]] \]

\[ = -[\bar{X}, [\bar{v}, \bar{w}]] = [\bar{v}, \bar{w}], \bar{X} \]

\[ = L [\bar{v}, \bar{w}] \bar{X} \]

The key step here has used the Jacobi identity.

b) Since \( L \bar{v} (\bar{f}) = \bar{v}(\bar{f}) \), we have done the scalar part of this in assignment 1. For vectors,

\[ [[L \bar{X}, L \bar{v}], L \bar{z}] \bar{v} = [L [\bar{X}, \bar{v}], \bar{z}] \bar{v} = L [[\bar{X}, \bar{v}], \bar{z}] \bar{v} \]

Since \( L \bar{X} \) is linear in \( \bar{X} \), the result follows once again from the Jacobi identity.
a) We have

\[ L_{\nabla} (f \nabla) := \nabla (f \nabla) - f \nabla \nabla = \nabla (f) \cdot \nabla + f \nabla \nabla - f \nabla \nabla = L_{\nabla} f \cdot \nabla + f \nabla \nabla \nabla \]

b) Here, we have

\[ L_{\nabla} \nabla = \nabla [\nabla, \nabla] = [\nabla e_i, \nabla e_j] = \nabla^i [\nabla e_i, \nabla e_j] - \nabla^j [\nabla e_j, \nabla e_i] = \nabla^i \nabla e_i + \nabla^j \nabla e_j - \nabla^j e_j (\nabla^i) e_i \]

The result follows immediately.

c) This follows from (2.7):

\[ (L_{\nabla} \nabla)^i = \delta^i_j \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} = - \frac{\partial}{\partial x^j} \delta^i_j = \frac{\partial}{\partial x^i} \]

We have noted that the only non-zero component of \( \nabla \) here is \( \nabla^i = 1 \).

By the Leibniz property

\[ (L_{\nabla} \tilde{w}) \cdot \nabla \tilde{w} = L_{\nabla} (\tilde{w} \cdot \nabla \tilde{w}) - \tilde{w} \cdot L_{\nabla} \nabla \tilde{w} = \nabla (\tilde{w} (\tilde{w})) - \tilde{w} (\nabla \nabla \tilde{w}) \]
Expressing things in components gives

\[(L \tilde{\omega})_i w^i = \nabla^i \partial_j (w_j w^i) - w_i (\nabla^i \partial_j w^i - w_j \partial_j \nabla^i)\]

\[= w_i \nabla^i \partial_j w_j + w_j w_i \partial_j \nabla^i\]

\[= w_i (\nabla^i \partial_j w_j + w_j \partial_j \nabla^i)\]

The result follows because \(\omega\) is arbitrary.

4. From (6.17) we have

\[L_x \omega^{bc} = x^m \partial_m \omega^{bc} + \omega^{mc} \partial_b \omega^{cm} + \omega^{bm} \partial_c \omega^{mb}\]

\[L_x (\gamma^a \omega^{bc}) = x^m \partial_m (\gamma^a \omega^{bc}) - (\gamma^m \omega^{bc}) \partial_m \omega^a + (\gamma^a \omega^{cm}) \partial_b \omega^c + (\gamma^a \omega^{bm}) \partial_c \omega^b\]

\[= \omega^{bc} (x^m \partial_m \gamma^a - \gamma^m \partial_m \omega^a) + \gamma^a (x^m \partial_m \omega^{bc} + \omega^{cm} \partial_b \omega^m + \omega^{bm} \partial_c \omega^m)\]

The Leibniz property is confirmed in the second equality.

5. a) This is obvious. The denominator \(1 \pm z\) vanishes only at \(z = \pm 1\). These points must therefore be excluded from the corresponding chart.

b) First, we must calculate \(\varphi^{-1}_s\):

\[(u, v) = \left(\frac{x}{1 - z}, \frac{y}{1 - z}\right) \rightarrow u^2 + v^2 = \frac{x^2 + y^2}{(1 - z)^2} = \frac{1 + z}{1 - z}\]
In the last equality, we have recalled that 
\(x^2 + y^2 + z^2 = 1\). Inverting this, we find

\[(1-z)(u^2 + v^2) = 1 + z \quad \Rightarrow \quad z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\]

\[
\Rightarrow \quad 1 - z = \frac{2}{u^2 + v^2 + 1}
\]

\[
\Rightarrow (x, y, z) = ((1-z)u, (1-z)v, z) = \left(\frac{2u}{u^2 + v^2 + 1}, \frac{2v}{u^2 + v^2 + 1}, \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1}\right)
\]

We now compose this with \(\Phi_N^\prime\):

\[
\Phi_N^\prime(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right) = (u', v')
\]

\[
1 + z = 1 + \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} = \frac{2(u^2 + v^2)}{u^2 + v^2 + 1}
\]

\[
\Rightarrow (u', v') = \left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2}\right)
\]

This map should be defined for all points except the two poles. Thus, \(u' = v' = 0\) (south pole) and \(u = v = 0\) (north pole) must be excluded. Except at the origin, the function \(\Phi_N \circ \Phi_N^{-1}(r) = \frac{r}{r^2}\) in two dimensions is clearly smooth.

c) This function also has a smooth inverse, which happens to be the same function from \(\mathbb{R}^2\) to \(\mathbb{R}^2\). Therefore, all of the overlap functions are smooth, and \(S^2\) is a manifold.

6 a) We need to show that \(S\) takes the same value for every \((z^1, z^2)\) on a given "line." But this is immediate: \(\frac{d(z^1)}{d(z^2)} = z^1/z^2\).
b) $S_1$ is defined unless $z^2 = 0$, meaning except on the line $[1, 0]$. Similarly, $S_2$ is defined except on $[0, 1]$.

c) If $w = S_1(z^2; z^2) = \frac{z^2}{z^2} + 4iz$

$$[w, 1] = \left[ \frac{z^2}{z^2}, 1 \right] = [z, z^2]$$

In the second equality, we have scaled both $z$'s by $z^2$, which doesn't change the line. The proof for $S_2$ is identical.

d) Both charts are defined unless $z^1 = 0$ or $z^2 = 0$.

Throwing out these points, we have

$$S_2 \circ S_1^{-1}(w) = S_2([w, 1]) = \frac{1}{w}$$

Since $S_1^{-1}(0) = [0, 1]$ is excluded, this mapping from $E$ to $E$ is analytic, and therefore smooth. The proof for $S_1 \circ S_2^{-1}$ is identical.

e) All overlap functions are analytic. This is done!

7 a) We have

$$[1+z, x+iy] = \left[ \frac{x-iy}{1+z}, \frac{x+iy}{1+z} \right] (x+iy)$$

$$= \left[ x-iy, \frac{x^2+iy^2}{1+z} \right]$$

$$= \left[ x-iy, \frac{1-z^2}{1+z} \right] = [x-iy, 1-z]$$

This is the result.
b) From the first expression, we have

\[ \frac{x+i}{1+z} = \frac{z^2}{z^1} \Rightarrow |\frac{z}{z^1}|^2 = \frac{x^2+y^2}{(1+z)^2} = \frac{1-z^2}{1+z^2} \]

\[ = \frac{z}{1+z^2} = \frac{1-z^2}{1+z^2} = \frac{z^2-1}{z^2+1} \]

\[ = x + iy = (1+z) \frac{z^2}{z^1} = \frac{z^2}{z^1} \frac{z^2}{z^1} = \frac{z}{1+z^2} \frac{z}{1+z^2} \]

Taking real and imaginary parts gives the result.

c) We have, for example

\[ \phi \circ \psi^{-1}_N (u, v) = \phi (\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{-u^2+v^2-1}{u^2+v^2+1}) \]

\[ = \psi_1 \left[ \frac{1 - \frac{u^2+v^2-1}{u^2+v^2+1}}{\frac{-u^2+v^2-1}{u^2+v^2+1}} \right] \]

\[ = \psi_1 \left[ \frac{2(u+i)}{u^2+v^2+1} (u+i) \right] = u + i \]

This is obviously smooth with smooth inverse. The other coordinate maps can be found by composing with the transition functions calculated above:

\[ \psi_2 \circ \phi \circ \psi^{-1}_N (u, v) = \frac{1}{u+i} \]

\[ \psi_1 \circ \phi \circ \psi^{-1}_S (u, v) = \frac{1}{u-i} \]

\[ \psi_2 \circ \phi \circ \psi^{-1}_S (u, v) = u - i \]

d) Since all of these maps are smooth wherever both charts are defined, the manifolds are diffeomorphic in the same sense that \( \mathbb{C} \) can be identified with \( \mathbb{R}^2 \).