Lecture 5
Gravity and Geometry

Who are the inertial observers?
How do they measure a non-trivial gravitational field?
Last time: Minkowski spacetime as seen by a uniformly accelerated observer:

\[ x(t) = \frac{c^2}{g} \cosh \left( \frac{g}{c} t \right) \]
\[ t(t) = \frac{c}{g} \sinh \left( \frac{g}{c} t \right) \]

\[ ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \]
\[ = e^{2g \frac{t^2}{c^2}} (-c^2 dt^2 + d\tilde{s}^2) + dy^2 + dz^2 \]
\[ = -(c^2 + g x) dt^2 + dx^2 + dy^2 + dz^2 \]

\[ \tilde{s} = \text{radio-distance to } E \]
\[ (\text{measured with clocks}) \]
\[ x = \text{metric distance to } E \]
\[ (\text{measured with rulers}) \]
Equivalence Principle

A uniformly accelerated observer experiences many of the effects we associate with gravity:

- Normal force
- Dropped objects fall

What does the free-fall curve look like to the accelerated observer?

\[ x(t) = \frac{c^2}{g} \sqrt{1 + \left(\frac{gt}{c}\right)^2} \]

\[ x(t) = \frac{c^2}{g} \]

\[ x = \frac{c^2}{g} e^{\frac{g^2 t^2}{c^2}} \cosh \frac{gt}{c} = \left(\frac{c^2}{g} + x\right) \cosh \frac{gt}{c} \]

\[ \frac{c^2}{g} \quad \text{we have} \quad x(\tau) = \frac{c^2}{g} \left( \text{sech} \frac{gt}{c} - 1 \right) = -\frac{1}{2} g \tau^2 + \frac{5}{24} \frac{g^3}{c^2} \tau^4 + ... \]
On sufficiently short length- and time-scales, uniformly accelerating observers in Minkowski spacetime are like observers in a uniform Newtonian gravitational field.

\[ \Delta x \ll \frac{c^2}{g} \quad \Delta t \ll \frac{c}{g} \]

Conversely, on sufficiently short length- and time-scales, a non-accelerating (no normal force) observer in a real gravitational field may believe herself to be in Minkowski spacetime.

Inertial observers are those in free fall!

(Equivalence Principle)
Gravity focuses free-fall observers.

Inertial observers are local: they must measure things on scales shorter than the focusing effect.
So, on short scales,
  - inertial (free-fall) observers move rectilinearly through Euclidean space at uniform relative speeds

  ⇒ Minkowski metric
  ⇒ Vector-space structure

On long scales,
  - inertial observers are focused and deflected by non-uniform gravitational fields.

  ⇒ non-Minkowski metric
  ⇒ no Vector-space structure

General relativity is a theory of a locally Minkowski metric

\[ ds^2 = \sum_{\alpha\beta} g_{\alpha\beta}(x) \, dx^\alpha \, dx^\beta \]

on a spacetime that is locally like \( \mathbb{R}^4 \) (a manifold)
Gravity and Geometry

Inertial observers move through locally Minkowski spacetime

\[ \frac{d}{d\tau} u^a = 0 \]

\[ u^a \quad \nabla_a u^b \propto u^b \]

"change of \( u^b \) as one moves along \( u^a \)"

How to measure focusing:
Consider two observers, each equipped with three gyroscopes:

Observer one follows $X$ then $Y$. Observer two follows $Y$ then $X$.

The gyroscopes keep track of the directions as they move, but respond to gravitational torques: $X^a \nabla^a Y^b = 0$

Knows about gravity
Facts about $\nabla_a$ (transport)

- gravity
  - inertial observers
    - spacetime metric
  - gravitational torques
    - spacetime transport

- apparently compatible

1. When moving vectors from one place to another in spacetime, their lengths and the angles between them (both measured with the local metric) appear to be preserved.
Torsion appears to be a second-order effect.

(Loops close, but transport of vectors around them may be non-trivial.)

Riemannian Geometry

- Metric tensor $g_{ab}$ measures infinitesimal lengths and angles.
- There is a unique transport operator $\nabla_a$ that is:
  - Metric-compatible: $\nabla_ag_{bc}=0$
  - Torsion-free.
- This $\nabla_a$ may have curvature
Riemann Normal Coordinates

Draw all non-accelerating curves from a given spacetime event.

Within a sufficiently small neighborhood, there are no conjugate points.

\Rightarrow there is a one-to-one map from initial velocity vectors to surrounding points of spacetime

\[ g_{ab}(x) = g_{ab}(\bar{x}) + R^c_{\ abd}(\bar{x}) \cdot (x-\bar{x})^c \cdot (x-\bar{x})^d + \mathcal{O}((x-\bar{x})^3), \]
So, what do we need?

- local vector space structure (differentiable manifolds)
- local Minkowski geometry (tensor fields)
- transport operation $\nabla_a$ (derivative operator/affine connection)
- characterize "focusing" (Riemann curvature)
- physics.

How do we compute a gravitational field from its source?

(Einstein field equation)
Geodesic Deviation

non-accelerating curves

\[ \frac{\ddot{\xi}^a}{\dot{\xi}} = (U^c \nabla_c) (U^b \nabla_b) \xi^a \]
\[ = -R_{c,bd} \xi^b U^c U^d \]

Riemann curvature is focusing of inertial observers!