Lecture 15

The Newtonian Limit
Newtonian Space-Time

Newtonian physics has a *universal* time variable $t$ and a preferred state of *absolute* rest, occupied by inertial observers with 4-velocity $u^a$.

The Newtonian limit of general relativity arises when sources are (a) weak and (b) move slowly enough that spacetime has approximately this structure. $(t, u^a)$
The Newtonian Limit

The Newtonian gravitational field is given by solving the Poisson equation

\[ \Delta \Phi = 4\pi \rho \]

for the gravitational potential \( \Phi \) in terms of the mass density \( \rho \).

To recover this limit from general relativity, we must assume that

- source speeds are slow (\( v \ll c \))
- we are close to the source (\( r \ll cT \))

(no retardation) dynamical time scale
In these limits, to order unity in the source speed, we may write

\[ T_{ab} \approx \rho_{\text{a}} \rho_{\text{b}} \]

mass \quad \text{absolute rest}

density

When we are close enough to ignore retardation, we may also neglect time-derivatives in the de-Donder-gauge post-Minkowski field equation:

\[ \Box h_{ab} \approx \Delta h_{ab} = -16\pi T_{ab} \]

Thus, we have the first-order post-Newtonian field equation

\[ \Delta h_{ab} = -16\pi \rho_{\text{a}} \rho_{\text{b}} \]
The vector field $t^a$ satisfies

$$\partial_a t^b = 0$$

and so commutes with the Laplacian on the Newtonian spatial slices ($t = \text{const.}$)

$$\Rightarrow h_{ab} = -4\Delta t_a t_b$$

with

$$\Delta \Xi = 4\Pi \rho \quad \text{Newtonian potential}$$

The metric perturbation is

$$h_{ab} = g_{ab} - \frac{1}{2} \gamma_{ab} \dot{g}$$

$$\Rightarrow h = g - \frac{1}{2} \cdot 4 \dot{g} = -\dot{g}$$

$$\Rightarrow \dot{g}_{ab} = h_{ab} + \frac{1}{2} \gamma_{ab} \dot{g}$$

$$= h_{ab} - \frac{1}{2} \gamma_{ab} h$$

$$= -4\Xi t_a t_b - \frac{1}{2} \gamma_{ab} - 4\Xi t_c t_c$$

$$= -2\Xi (2t_a t_b + \gamma_{ab})$$
To leading order, the physical metric is therefore

\[ g_{ab} = \gamma_{ab} + \delta g_{ab} \]

\[ = (-t_a t_b + \delta_{ab}) - 2\Xi (t_a t_b + \delta_{ab}) \]

\[ = -(1 + 2\Xi) t_a t_b + (1 - 2\Xi) \delta_{ab} \]

spatial metric on Newtonian slices.

Note: We have built the perturbation parameter \( \varLambda \) into the potential here:

\[ \varPi = \frac{\text{potential energy}}{\text{unit mass}} \sim \frac{E}{M} \sim c^2 \]

\[ ds^2 = -(c^2 + 2\Xi) dt^2 \]

\[ + (1 - \frac{2\Xi}{c^2}) (dx^2 + dy^2 + dz^2) \]

For the field outside a star, e.g., we must have \( \frac{G M^*}{c^2 R^*} \ll 1 \)
Motion of Test Bodies

Working in the background "Newtonian" inertial coordinates, the geodesic equation is

$$u^a \nabla_a u^b = 0$$

$$\Rightarrow \frac{d^2 x^B}{d\tau^2} - \Gamma^B_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0$$

For slowly moving test masses, we may write, to leading order,

$$\frac{dx^\alpha}{d\tau} \sim t^\alpha \quad \text{and} \quad d\tau \sim dt$$

Thus, we have

$$\frac{d^2 x^B}{d\tau^2} = \Gamma_{00}^B = -\frac{1}{2} g^{B\alpha} (2 \delta(0) g_{00} \alpha$$

$$= \frac{1}{2} \delta^B (-c^2 - 2\phi)$$

$$= -\delta^B \Phi \quad \leftarrow \frac{\dot{\chi}}{\chi} = -\Phi$$
Deflection of Light

Light is not affected by gravity in Newtonian physics, but it follows geodesics even in post-Minkowskian gravity.

Q: How does the gravitational field of the sun affect light from distant stars?

A: We must repeat the previous calculation for a null geodesic.
Set up the problem as follows:

\[ \tan \phi = \frac{K^y_{+}}{K^x_{+}} \]

Let \( \lambda \) be an affine parameter along the null geodesic:

\[ K^a = \frac{dx^a}{d\lambda} \quad \text{and} \quad \frac{dK^a}{d\lambda} = \Gamma^a_{\mu\nu} K^\mu K^\nu \]

\[ \Rightarrow \frac{dK^y}{d\lambda} = \Gamma^y_{tt} K^t K^t + \Gamma^y_{xx} K^x K^x \]

(Note: \( \Gamma^a_{\mu\nu} \) is already first-order, so we can use the zeroth-order approximants \( K^x = K^t \)).
\[ \Gamma_{tt} Y = -\frac{1}{2} g^{tt} Y (2 \partial_t (t g_{tt}) Y - \partial_t g_{tt}) = \frac{1}{2} \partial_t Y (-c^2 - z \Phi) = -\partial_t Y \Phi \]

\[ \Gamma_{xx} Y = -\frac{1}{2} g^{xx} Y (2 \partial_x (x g_{xx}) Y - \partial_x g_{xx}) = \frac{1}{2} \partial_x Y (1 - z \Phi) = -\partial_x Y \Phi \]

\[ \Rightarrow \frac{d K^Y}{d \lambda} = -2 \partial Y \Phi \cdot K^x K^x \]

Now let's take the potential

\[ \Phi = -\frac{M}{r} \]

outside the sun:

\[ \frac{d \Phi}{d Y} = \frac{M Y}{r^3} \approx \frac{-M b}{(x^2 + b^2)^{3/2}} \]

\[ \Rightarrow \frac{d K^Y}{d \lambda} = \frac{2 M b \cdot K^x}{(x^2 + b^2)^{3/2}} \frac{d x}{d \lambda} \]

\[ \Rightarrow \Delta K^Y = \frac{2 M b K^x}{b^2 \sqrt{x^2 + b^2}} \Delta \frac{x}{b^2 \sqrt{x^2 + b^2}} \]

\[ = \frac{4M}{b} K^x \]
This gives the deflection

\[ \varphi \equiv \tan \varphi \equiv \frac{4M}{b} \]

For example, for the sun, we can calculate

\[ \frac{4M \theta}{R_0} = \frac{4G M_0}{c^2 R_0} \]

\[ = \frac{4(6.67 \times 10^{-8})(1.99 \times 10^{33})}{(3.00 \times 10^{10})^2 (6.96 \times 10^{10})} \]

\[ = 8.48 \times 10^{-6} \text{ (radians)} \]

\[ = 1.75'' \]

Thus, we find the deflection

\[ \varphi_0 = (1.75'') \frac{R_0}{b} \]

observed by Eddington.
Gravitational Radiation

Suppose we continue to work with weak, slow-motion sources, but look at large distances where we cannot ignore retardation effects:

\[ h_{ab} = -16\pi T_{ab} \]

\[ h_{ab}(t, \vec{x}) = 4\int \frac{T_{ab}(t - 1/\gamma \vec{x}/\gamma, \vec{y})}{1/\gamma \vec{x}/\gamma} \, d^3y \]

\[ \downarrow \quad \text{integral over} \]
\[ \text{background} \quad \text{past light} \]
\[ \text{inertial components} \quad \text{cone} \]

Retarded scalar Green function:

\[ G_y(x) = -\frac{1}{2\pi} \Theta_{ty}(t_x) \delta(\|x - y\|^2) \]

\[ = -\frac{1}{4\pi} \frac{\delta(t_y - t_x + 1x/\gamma)}{1x/\gamma} \]
Fourier transform in $t$

\[
\hat{h}_{a,b}(w,x) = 4 \sqrt{\frac{\hat{f}_{a,b}(w,y)}{|x-y|}} e^{iwx} d^3y
\]

Look in the "wave zone"

$wR \gg 1$

\[
\hat{h}_{a,b}(w,x) = 4 \frac{e^{iwx}}{R} \int \hat{f}_{a,b}(w,y) d^3y
\]