Lecture 10

Torsion and Curvature
Parallel Transport

To do differential calculus with vector fields, we must take derivatives of one vector field along another: $\nabla_v W^a$

The Lie derivative does not let us do this!

1. $L_v W^a = [V, W]^a$

2. $[V, W](t) = V(w(t)) - W(V(t))$

3. $V(w(t)) = V^\alpha \partial_\alpha (w^B \partial_B (t))$

$$= V^\alpha \partial_\alpha (w^B) \partial_B (t) + V^\alpha w^B \partial_\alpha \partial_B (t)$$

$$= [V, W](t) = [V^\alpha w^B - w(V^B)] \partial_B (t)$$

$$+ z V^\alpha w^B \partial_\alpha \partial_B (t)$$

$s = 0$
\( L_v W^a \) requires two vector fields. It does not let us take the derivative of a vector field \( W^a \) along a single integral curve of \( V^a \).

We need an additional structure \( \nabla_j W^a \) to take the derivative of \( W^a(p) \) as we move along \( \gamma(t) \).

This should take derivatives of \( W^a \), but be \textbf{algebraic} in \( j^a \).

\[
\nabla_f V^a = f \nabla V^a \quad \text{function}
\]

\[
\nabla V(f W^a) = f \nabla V W^a + W^a \nabla V f
\]

Naturally, we define \( \nabla_V f := V(f) \).
Derivative Operators
(Covariant Derivative, Affine Connection)

Given: vector $W^a$ and vector field $V^a$

- $\nabla w V^a$ is functionally linear in $W^b$ (⇔ algebraic)
- $\nabla w V^a$ is linear in $V^a$
- $\nabla w f = W(f)$
- $\nabla w T$ is linear and Leibniz on tensor fields
- $\nabla w \delta_a^b = 0$

Example: Coordinate Derivative

$\partial_w V^a = \partial_w (V^a \delta^b_a) = W(V^a) \, b_a^b$

Transport: $\partial_w V^a \equiv \partial_{\alpha} \Rightarrow \text{keep the components constant!}$
The Space of Derivative Operators

Notation: $\nabla^a V^a =: W^b \nabla_b V^a$

emphasizes functional linearity in $W^b$

Let $\tilde{\nabla}_a$ and $\nabla_a$ denote two derivative operators:

$$(\tilde{\nabla}_a - \nabla_a)(fW_b) = \tilde{\nabla}_a(fW_b) - \nabla_a(fW_b)$$
$$= W_b \tilde{\nabla}_a f + f \tilde{\nabla}_a W_b$$
$$- W_b \nabla_a f - f \nabla_a W_b$$
$$= W_b [(df)_a - (df)_a] + f(\tilde{\nabla}_a - \nabla_a)W_b$$

$\Rightarrow (\tilde{\nabla}_a - \nabla_a)W_b$ is functionally linear in $W_b$ (algebraic)

$\Rightarrow (\tilde{\nabla}_a - \nabla_a)W_b = \epsilon_{abc} W_c$

← tensor!
Example: Christoffel Symbols

\[ \nabla_a = \text{connection} \]
\[ \delta_a = \text{coordinate connection} \]

\[
(nabla_a - \delta_a) w_b = \Gamma_{abc} w_c
\]

\text{Christoffel tensor}

\[ \tilde{\nabla}_a = \text{another coordinate connection} \]

\[
(nabla_a - \tilde{\delta}_a) w_b = \tilde{\Gamma}_{abc} w_c
\]

\text{another Christoffel tensor}

\[
\tilde{\Gamma}_{abc} w_c = \Gamma_{abc} w_c + (\delta_a - \tilde{\delta}_a) w_b
\]

\text{"non-tensorial" term}

The Christoffel symbols \( \Gamma_{abc} \) do not transform like a tensor because they are different tensors!
The set of all derivative operators on a manifold $M$ is naturally an affine space!

Given $\nabla_a$ and $\tilde{\nabla}_a$, define

$$[\lambda \nabla_a + (1-\lambda) \tilde{\nabla}_a] \gamma_\alpha = \lambda \nabla_a \gamma_\alpha + (1-\lambda) \tilde{\nabla}_a \gamma_\alpha = (d\gamma_\alpha)_a$$

$\cdot [\lambda \nabla_a + (1-\lambda) \tilde{\nabla}_a] \delta_\beta = 0$, etc.

We can draw straight lines in the space of derivative operators, but there is no natural origin $\tilde{\nabla}_a$.

The connection is a physical field.
How do the actions of two derivative operators differ on other tensor fields?

Use the Leibniz property:

$$(\tilde{\nabla}_a - \nabla_a) V^b \cdot \omega_b$$

$$= \omega_b \tilde{\nabla}_a V^b - \omega_b \nabla_a V^b$$

$$= \tilde{\nabla}_a (\omega_b V^b) - V^b \tilde{\nabla}_a \omega_b$$

$$= V^b (\tilde{\nabla}_a - \nabla_a) \omega_b$$

$$= V^b C_{abc} \omega_c$$

$$= V^c C_{abc} \omega_b = -C_{abc} V^c \cdot \omega_b$$

$$(\tilde{\nabla}_a - \nabla_a) V^b = -C_{abc} V^c$$

$$\omega^b (\tilde{\nabla}_a - \nabla_a) T_{b_1 \ldots b_m}^{c_1 \ldots c_n}$$

$$= \sum_{i=1}^{m} C_{abc} \omega^{b_i} T_{b_1 \ldots d \ldots b_m}^{c_1 \ldots c_n}$$

$$- \sum_{j=1}^{n} C_{acd} \omega^{c_j} T_{b_1 \ldots a \ldots b_m}^{c_1 \ldots e \ldots c_n}$$
Torsion

Let $\nabla$ be a derivative operator, and define the bracket

$$\left[ v, w \right]_{\nabla} := \nabla_v w - \nabla_w v$$

of vector fields.

This bracket is not functionally linear in either argument:

$$\left[ v, f w \right]_{\nabla} = \nabla_v (f w) - \nabla_{f w} v$$

$$= \nabla_v f \cdot w + f \nabla_v w - f \nabla_w v$$

$$= \nabla(f \cdot w) + f \left[ v, w \right]_{\nabla}$$

The ordinary Lie bracket has the same behavior

$$\left[ v, f w \right](g) = \nabla(f w(g)) - f w(v(g))$$

$$= \nabla(f \cdot w(g)) + f \nabla(w(g)) - f w(v(g))$$

$$\Rightarrow \left[ v, f w \right] = \nabla(f \cdot w) + f \left[ v, w \right]$$
Neither $[v, w]_\alpha$ nor $[v, w]$ depends algebraically on the values of $V$ and $W$ at a point, but their difference does:

$$[v, w]_\alpha - [v, w] = T(v, w)$$

A linear map taking two vectors to one $\mapsto (\cdot)$ tensor field.

$\mapsto$ Torsion tensor $T^c_{ab}$

The torsion tensor is necessarily anti-symmetric $T_{(a b)}^c = 0$ because both brackets are.
\[ T_{abc} = x^a y^b w^c \]
\[ \nabla_w (x^a y^b w^c) = y^b w^c \nabla_w x^a + x^a w^c \nabla_w y^b + x^a y^b \nabla_w w^c \]
\[ \tilde{T}(v, w) - T(v, w) \]
\[ = [v, w] - [v, w] \]
\[ - [v, \dot{w}] + [v, \dot{w}] \]
\[ = \tilde{\nabla}_v w - \tilde{\nabla}_w v - \nabla_v w + \nabla_w v \]

\[ \tilde{T}_{ab}^c - T_{ab}^c = -C_{ab}^c + C_{ba}^c \]
\[ = -2C_{[ab]}^c \]

\[ V^a(\tilde{\nabla}_a - \nabla_a)w^c - W^a(\tilde{\nabla}_a - \nabla_a)V^c \]
\[ = -V^aC_{ab}^c w^b + W^aC_{ab}^c V^b \]
\[ = -V^a w^b (C_{ab}^c - C_{ba}^c) \]
The torsion tensor is also related to the commutator of covariant derivatives of functions.

\[ v^a w^b T_{ab}^c \partial_c f \]

\[ = (\partial_v w - \partial_w v - [v, w])^c \partial_c f \]

\[ = (\partial_v w^c - \partial_w v^c - [v, w]^c)^c \partial_c f \]

\[ = \partial_v (w^c \partial_c f) - w^c \partial_v \partial_c f \]

\[ = \partial_v (w^c \partial_c f) - w^c \partial_v \partial_c f \]

\[ = \partial_v (w^c \partial_c f) - w^c \partial_v \partial_c f \]

\[ = v^c w^d \partial_d \partial_c f - w^c v^d \partial_d \partial_c f \]

\[ = -2 v^c w^d \partial_c \partial_d f \]

\[ 2 \partial_{[a} \partial_{b]} f = -T_{ab}^c \partial_c f \]