Structure of Minkowski Space

• There are five kinds of vectors in Minkowski spacetime:

  • space-like $\|\mathbf{x}\|^2 > 0$
  • time-like $\|\mathbf{x}\|^2 < 0$
  • and light-like. $\|\mathbf{x}\|^2 = 0$

• Time-like and light-like vectors can be either

  • future-directed $t_x > 0$
  • or past-directed. $t_x < 0$

\[
\|\mathbf{x}\|^2 := -c^2 t_x^2 + |\vec{x}|^2
\]
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\[
\begin{align*}
\|\mathbf{x}\|^2 &:= -c^2 t_\mathbf{x}^2 + |\vec{x}|^2 \\
t' &= \gamma (t - \vec{v} \cdot \vec{x} / c^2) \\
|\vec{v} \cdot \vec{x}| &\leq |\vec{v}| |\vec{x}| < vc |t_\mathbf{x}| < c^2 |t_\mathbf{x}|
\end{align*}
\]
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\[ t' = \gamma (t - \mathbf{v} \cdot \mathbf{x}/c^2) \]
\[ |\mathbf{v} \cdot \mathbf{x}| \leq |\mathbf{v}| |\mathbf{x}| < vc |t_x| < c^2 |t_x| \]
\[ \|\mathbf{x}\|^2 := -c^2 t_x^2 + |\mathbf{x}|^2 \]
Kinematical Effects
Length Contraction

An inertial observer $O'$ carries a ruler of length $L_0$ at speed $v$ past an inertial observer $O$.

How long does $O$ measure it to be?
Length Contraction

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How long does $O$ measure it to be?

How long does $O'$ measure and identical ruler carried by $O$ to be?

$$x_0' = 0 = \gamma (x_0 - vt)$$

$$x_1' = L_0 = \gamma (x_1 - vt)$$

$$x_1(t) - x_0(t) = \frac{L_0}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}} L_0$$
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$$x_1(t) - x_0(t) = \frac{L_0}{\gamma} = \sqrt{1 - v^2/c^2} L_0$$

$$x_0 = 0 \quad x' = \gamma (x - vt)$$

$$x_1 = L_0 \quad t' = \gamma (t - vx/c^2)$$

$$x' + vt' = \gamma (x - v^2 x/c^2)$$
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Length of $O'$ ruler measured by $O = ||A|| < ||B||$

Length of $O$ ruler measured by $O' = ||C|| < ||D||$

\[
x_1(t) - x_0(t) = \frac{L_0}{\gamma} = \sqrt{1 - \frac{v^2}{c^2}} L_0
\]

\[
x'_1(t') - x'_0(t') = \frac{L_0}{\gamma}
\]
Time Dilation

An inertial observer $O'$ carries a clock that advances a time $T_0$ while she passes $O$ at speed $v$.

How much time elapses for $O$?

What happens if the roles are reversed?

$$T := t_E = \gamma T_0 = \frac{T_0}{\sqrt{1 - v^2/c^2}}$$

$$\vec{x}_{O'}(t) = \vec{v} t$$

$$t'_E = \gamma (t_E - \vec{v} \cdot \vec{x}_E/c^2) = \gamma (1 - v^2/c^2) t_E = \frac{t_E}{\gamma}$$
Time Dilation

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How much time elapses for $O$?

What happens if the roles are reversed?

\[
T := t_E = \gamma T_0 = \frac{T_0}{\sqrt{1 - v^2/c^2}}
\]

\[
T' = \gamma T_0
\]

\[
\vec{x}_{O'}(t) = \vec{v} t
\]

\[
t'_{G} = \gamma (t_G - \vec{v} \cdot \vec{x}_G/c^2) = \gamma T_0
\]
Time Dilation

An inertial observer \( O' \) carries a clock that advances a time \( T_0 \) while she passes \( O \) at speed \( v \).

How much time elapses for \( O \)?

What happens if the roles are reversed?

\[
T := t_E = \gamma T_0 = \frac{T_0}{\sqrt{1 - v^2/c^2}}
\]

\[
T' = \gamma T_0
\]

\[
\vec{x}_{O'}(t) = \vec{v}t
\]

\[
\sqrt{-\|F\|^2} > \sqrt{-\|G\|^2} = \sqrt{-\|E\|^2} < \sqrt{-\|H\|^2}
\]
Velocity Addition

A particle moves with uniform velocity \( u' \) relative to \( O' \), who moves with uniform velocity \( v \) relative to \( O \).

This particle will move with uniform velocity \( u \) relative to \( O \). What is it?

\[
[I + (\gamma - 1) \hat{v} \hat{v} + \vec{u}' \vec{v} / c^2]^{-1} = I - \frac{(1 - \gamma^{-1}) \hat{v} \hat{v}}{1 + \vec{v} \cdot \vec{u}' / c^2} - \frac{\vec{u}' \vec{v} / c^2}{1 + \vec{v} \cdot \vec{u}' / c^2}
\]

Exercise!

\[ \vec{x}' = \vec{u}' \cdot t' \]

\[ [I + (\gamma - 1) \hat{v} \hat{v}] \cdot \vec{x} - \gamma \vec{v} \cdot t = \vec{u}' \gamma \left( t - \vec{v} \cdot \vec{x} / c^2 \right) \]

\[ [I + (\gamma - 1) \hat{v} \hat{v} + \vec{u}' \vec{v} / c^2] \cdot \vec{x} = \gamma (\vec{v} + \vec{u}') \cdot t \]

\[ \vec{x} = \vec{u} \cdot t \quad \text{with} \quad \vec{u} = \frac{\vec{v} + \vec{u}'_\parallel + \vec{u}'_\perp / \gamma}{1 + \vec{v} \cdot \vec{u}' / c^2} \]
Aberration

How does the angle of incidence of a stream of particles depend on the motion of the observer?

How does the angle of incidence of a wave depend on the motion of the observer?
Aberration

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Aberration

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How does the angle of incidence of a wave depend on the motion of the observer?

\[
\frac{u'^2}{c^2} = 1 - \frac{(1 - v^2/c^2)(1 - u^2/c^2)}{[1 + (vu/c^2) \cos \alpha]^2}
\]

\[
\tan \alpha' = \frac{\sin \alpha}{\gamma (\cos \alpha + v/u)}
\]
Aberration

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\tan \alpha' = \frac{\sin \alpha}{\gamma (\cos \alpha + v/u)}
\]

\[
\Psi(t, \vec{x}) = A e^{-i(\omega t - \vec{k} \cdot \vec{x})}
\]

\[
0 = \frac{d}{dt} (\omega t - \vec{k} \cdot \vec{x}) = \omega - \vec{k} \cdot \vec{u}
\]

\[
\omega t - \vec{k} \cdot \vec{x} = \omega \gamma (t' + \vec{v} \cdot \vec{x}'/c^2) - \vec{k} \cdot [\vec{x}' + (\gamma - 1) \hat{\nu} \nu \cdot \vec{x}' + \gamma \vec{u} t]
\]
(Particle Case)

$$\tan \alpha' = \frac{\sin \alpha}{\gamma (\cos \alpha + v/u)}$$

$$\frac{u'^2}{c^2} = 1 - \frac{(1 - v^2/c^2) (1 - u^2/c^2)}{[1 + (vu/c^2) \cos \alpha]^2}$$

**Aberration**

How does the angle of incidence of a stream of particles depend on the motion of the observer?

How does the angle of incidence of a wave depend on the motion of the observer?

$$\Psi(t, \vec{x}) = A e^{-i(\omega t - \vec{k} \cdot \vec{x})}$$

$$0 = \frac{d}{dt} (\omega t - \vec{k} \cdot \vec{x}) = \omega - \vec{k} \cdot \vec{u}$$

$$\left( \begin{array}{c} \omega'/c^2 \\ \vec{k}' \end{array} \right) = \left( \begin{array}{cc} \gamma & -\gamma c^{-2} \vec{v} \cdot \\ -\gamma \vec{v} & I \cdot + (\gamma - 1) \hat{v} \hat{v} \cdot \end{array} \right) \left( \begin{array}{c} \omega/c^2 \\ \vec{k} \end{array} \right)$$

$$\omega t - \vec{k} \cdot \vec{x} = \gamma (\omega - \vec{v} \cdot \vec{k}) t' - [\vec{k} + (\gamma - 1) \vec{k} \cdot \hat{v} \hat{v} - \gamma \vec{v} \omega/c^2] \cdot \vec{x}'$$
(Particle Case)
\[
\tan \alpha' = \frac{\sin \alpha}{\gamma (\cos \alpha + v/u)}
\]
\[
\frac{u'^2}{c^2} = 1 - \frac{(1 - v^2/c^2) (1 - u^2/c^2)}{[1 + (vu/c^2) \cos \alpha]^2}
\]

**Aberration**

How does the angle of incidence of a stream of particles depend on the motion of the observer?

How does the angle of incidence of a wave depend on the motion of the observer?

\[
\omega' = \gamma (\omega + vk \cos \alpha)
\]
\[
k' \cos \alpha' = \gamma (k \cos \alpha + v\omega/c^2)
\]
\[
k' \sin \alpha' = k \sin \alpha
\]
Aberration

How does the angle of incidence of a stream of particles depend on the motion of the observer?

How does the angle of incidence of a wave depend on the motion of the observer?

(Particle Case)

\[
\frac{u'}{c^2} = 1 - \frac{(1 - v^2/c^2)(1 - u^2/c^2)}{[1 + (vu/c)^2 \cos \alpha]^2}
\]

\[
\tan \alpha' = \frac{\sin \alpha}{\gamma (\cos \alpha + v/u)}
\]

(Wave Case)

\[
\frac{c^2}{u'^2} = 1 - \frac{(1 - v^2/c^2)(1 - c^2/u^2)}{[1 + (v/u) \cos \alpha]^2}
\]

\[
\tan \alpha' = \frac{\sin \alpha}{\gamma (\cos \alpha + vu/c^2)}
\]
**Aberration**

How does the angle of incidence of a stream of particles depend on the motion of the observer?

How does the angle of incidence of a **wave** depend on the motion of the observer?

\[
\tan \alpha' = \frac{\sin \alpha}{\gamma (\cos \alpha + v/u)}
\]

\[
\frac{u'^2}{c^2} = 1 - \frac{(1 - v^2/c^2) (1 - u^2/c^2)}{[1 + (vu/c^2) \cos \alpha]^2}
\]

\[
u = \frac{v - \hat{k} \hat{k} \cdot v}{\sqrt{1 + (\hat{k} \cdot v/c)^2}}
\]
Doppler Shift

Suppose a moving source emits pulses of light periodically at frequency $\omega_0$.

With what frequency does an inertial observer $O$ see pulses?

At what time $t$ does the pulse emitted at time $s$ arrive at $O$?

$$t(s) = s + \frac{\left| \vec{r}'(s) \right|}{c}$$

$$\dot{t}(s) = 1 + \frac{\vec{r}'(s) \cdot \vec{r}(s)}{c \left| \vec{r}(s) \right|} = 1 + \frac{\hat{r}(s) \cdot \vec{r}(s)}{c}$$

$$\Delta t \approx \left( 1 + \frac{\hat{r}(s) \cdot \vec{v}}{c} \right) \gamma(s) \Delta s_0$$

$$\frac{\omega_0}{\omega} = \left[ \frac{1 + \hat{r} \cdot \vec{v}/c}{\sqrt{1 - v^2/c^2}} \right]_{\text{ret}}$$

$$\ddot{t}(s) = \frac{\vec{r}'(s) \cdot \vec{r}'(s) + \vec{r}'(s) \cdot \vec{r}''(s)}{c \left| \vec{r}'(s) \right|} - \frac{(\vec{r}'(s) \cdot \vec{r}'(s))^2}{c \left| \vec{r}'(s) \right|^3} = \frac{\hat{r}(s) \cdot \vec{r}'(s)}{c} + \frac{\vec{r}'(s) \cdot \left[ I - \hat{r}(s) \hat{r}(s) \right] \cdot \vec{r}'(s)}{c \left| \vec{r}'(s) \right|}$$