Problem Set III
Due: Tuesday, 28 September 2010

• Do (at least) four of the following five problems from the text.
• Solutions are due (no later than) at the beginning of class.

1. (Exercise B.1, p. 790)
Suppose that we exponentially suppress high frequencies by multiplying the Fourier amplitude \( \tilde{f}(k) \) by \( e^{-\epsilon |k|} \). Show that the original signal \( f(x) \) is smoothed by convolution with a Lorentzian approximation to the delta function

\[
\delta_L^\epsilon (x - \xi) = \frac{1}{\pi \epsilon^2 + (x - \xi)^2}.
\]

As \( \epsilon \to 0 \), observe that \( \delta_L^\epsilon (x) \to \delta(x) \) in the sense of distributions.

2. (Exercises B.3, p. 790, and B.6, p. 792, the Hilbert transform)
   a. Show that the sum

   \[
   D_r(\theta) := \sum_{n=-\infty}^{\infty} \text{sgn}(n) e^{in\theta} r^{|n|} = \frac{r e^{i\theta}}{1 - r e^{i\theta}} - \frac{r e^{-i\theta}}{1 - r e^{-i\theta}},
   \]

   which converges for \( 0 < r < 1 \), approaches the principal-value distribution

   \[
   D(\theta) := i \mathcal{P} \cot \frac{\theta}{2}
   \]

   in the limit \( r \to 1 \).
   b. Let \( f(\theta) \) be a smooth function on the unit circle and define its Hilbert transform

   \[
   \mathcal{H} f(\theta) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta') \cot \left(\frac{\theta - \theta'}{2}\right) d\theta'.
   \]

   Show that \( f(\theta) \) can be recovered if one knows both its Hilbert transform \( \mathcal{H} f(\theta) \) and its average value \( \langle f \rangle \), according to the formula

   \[
   f(\theta) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{H} f(\theta') \cot \left(\frac{\theta - \theta'}{2}\right) d\theta' + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta') d\theta' =: -\mathcal{H}^2 f(\theta) + \langle f \rangle.
   \]
   c. Let \( f(x) \) be a function on the real line such that \( \int_{-\infty}^{\infty} |f(x)| dx \) is finite. Take a suitable limit in the previous result to show that \( \mathcal{H}^2 f(x) = -f(x) \), where

   \[
   \mathcal{H} f(x) := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(x')}{x - x'} dx'
   \]

   defines the Hilbert transform of a function on the real line.
3. (Exercises 2.3, p. 64, and 2.5, p. 65)

   a. Evaluate the integral

   \[ F(s, t) = \int_{-\infty}^{\infty} e^{-x^2} e^{2sx} e^{2tx} \, dx \]

   and expand both sides of your result as double power series in \( s \) and \( t \). By comparing the
   coefficients of \( s^m t^n \) on either side, show that

   \[ \int_{-\infty}^{\infty} H_m(x) H_n(x) e^{-x^2} \, dx = 2^n n! \sqrt{\pi} \delta_{mn}. \]

   b. Define the \textbf{normalized Hermite functions}

   \[ \varphi_n(x) := \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(x) e^{-x^2/2} \]

   and the \textbf{Fourier transform operator}

   \[ \mathcal{F} f(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixs} f(s) \, ds. \]

   Note that \( \mathcal{F}^4 \) is the identity map when the integral is normalized in this way, whence the
   only possible eigenvalues of \( \mathcal{F} \) are \( \pm 1 \) and \( \pm i \). Starting from Eq. (2.56) of the book, or
   otherwise, show that \( \varphi_n(x) \) is an eigenfunction of \( \mathcal{F} \) with eigenvalue \( i^n \).

4. (Exercise 2.13, p. 78)

   The completeness of a set \( \{P_n(x)\} \) of polynomials that are orthonormal with respect to a
   positive weight function \( w(x) \) may be expressed mathematically in the form

   \[ \sum_{n=0}^{\infty} P_n(x) P_n(y) = \frac{\delta(x-y)}{w(x)}. \]

   It is sometimes useful to have a formula for the partial sums of this infinite series. Suppose that
   the \( P_n(x) \) obey the three-term recurrence relation

   \[ x P_n(x) = b_n P_{n+1}(x) + a_n P_n(x) + b_{n-1} P_{n-1}(x), \]

   subject to the initial conditions

   \[ P_{-1}(x) = 0 \quad \text{and} \quad P_0(x) = 1. \]

   Use this recurrence relation, together with its initial conditions, to obtain the \textbf{Christoffel–Darboux partial sum} formula

   \[ \sum_{n=0}^{N-1} P_n(x) P_n(y) = b_{N-1} \frac{P_N(x) P_{N-1}(y) - P_{N-1}(x) P_N(y)}{x-y}. \]
5. (Exercises 2.20, 2.21 and 2.22, p. 84)

a. Let $f(x)$ be a continuous function. Observe that $f(x) \delta(x) = f(0) \delta(x)$ to deduce that

$$\frac{d}{dx} [f(x) \delta(x)] = f(0) \delta'(x).$$

If $f(x)$ is not only continuous but differentiable, then we can use the product rule to compute the above derivative in the form

$$\frac{d}{dx} [f(x) \delta(x)] = f'(x) \delta(x) + f(x) \delta'(x).$$

Show that these two expressions are equivalent in the sense of distributions by integrating the right side of each against an arbitrary test function $\varphi(x)$.

b. Let $\varphi(x)$ be a test function. Show that

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \frac{\mathcal{P}}{x-t} \varphi(x) \, dx = \int_{-\infty}^{\infty} \frac{\varphi(x) - \varphi(t)}{(x-t)^2} \, dx.$$

Show further that the right-hand side of this equation is equal to

$$-\left( \frac{\partial}{\partial x} \frac{\mathcal{P}}{x-t}, \varphi \right) := \int_{-\infty}^{\infty} \frac{\mathcal{P}}{x-t} \varphi'(x) \, dx.$$

c. Let $\theta(x)$ denote the step function or Heaviside distribution

$$\theta(x) := \begin{cases} 1 & x > 0 \\ \text{undefined} & x = 0 \\ 0 & x < 0. \end{cases}$$

Derive the equation

$$\lim_{\epsilon \to 0^+} \ln(x + i\epsilon) = \ln|x| + i\pi \theta(-x),$$

and take the weak derivative of both sides to show that

$$\lim_{\epsilon \to 0^+} \frac{1}{x + i\epsilon} = \frac{\mathcal{P}}{x} - i\pi \delta(x).$$