Review Problems V

Recommended Reading: Chow, pp. 100–121.

1. (Chow 3.4) Given the matrices

\[ A := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad C := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \]

show that \( A \) and \( B \) commute, that \( B \) and \( C \) commute, but that \( A \) and \( C \) do not.

2. (Chow 3.7) Show that the matrix

\[ A := \begin{pmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{pmatrix} \]

is not invertible.

3. (Chow 3.9) Find the inverse of the matrix

\[ A := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix} \]

and check your result by direct calculation.

4. (based on Chow 3.15) Let \( A \) and \( B \) be an arbitrary matrices whose product \( AB \) exists.

a. Show that the products \( AA^\dagger \) and \( A^\dagger A \) both exist and are Hermitian. Are these products equal to one another?

b. Show that the product \( B^\dagger A^\dagger \) exists and is equal to \((AB)^\dagger\).

c. If \( A \) and \( B \) are both Hermitian, then show that \( AB + BA \) is Hermitian as well.

d. If \( A \) and \( B \) are both Hermitian, then show that \( i(AB - BA) \) is Hermitian as well.
5. (based on Chow 3.19) The **Pauli spin matrices**

\[
\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

play an important role in quantum mechanics.

a. Show that each of these matrices is both Hermitian and unitary. Calculate the inverse of each.

b. Show that the product of two Pauli matrices is

\[
\sigma_i \sigma_j = \delta_{ij} I + i \sum_k \epsilon_{ijk} \sigma_k,
\]

where \( \delta_{ij} \) and \( \epsilon_{ijk} \) are the Kronecker and Levi-Civita symbols, respectively.

c. Calculate the commutator \([\sigma_i, \sigma_j]\) of two Pauli matrices.

6. (Chow 3.21) Let \( A, B \) and \( C \) be square matrices of the same dimension. Show that

\[
\text{tr} AB = \text{tr} BA \quad \text{and} \quad \text{tr} ABC = \text{tr} BCA = \text{tr} CAB.
\]

Is \( \text{tr} ACB = \text{tr} ABC \)?