Review Problems I

Recommended Reading: Chow, Appendix 1; Boas, Chapter 1.

1. (Chow A1.7, p. 510) Prove that
\[ \sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}, \]
and that
\[ A \cos x + B \sin x = \sqrt{A^2 + B^2} \sin(x + \delta), \quad \text{where} \quad \tan \delta := \frac{A}{B} \]
for all real \( x, A \) and \( B \).

2. (Chow A1.8, p. 510) Prove that
\[ \cosh^2 x - \sinh^2 x = 1 \quad \text{and} \quad \text{sech}^2 x + \tanh^2 x = 1 \]
for all real \( x \).

3. (Chow A1.9, p. 511) Calculate the limit of the function \( f(x) := x^2 \) as \( x \to 2 \), and show that \( f(x) \) is continuous there.

4. (Boas 1.4.6) Evaluate the partial sums
\[ S_N := \sum_{n=1}^{N} \frac{1}{n(n+1)}, \]
and show that they converge in the limit \( N \to \infty \).

Hint: Expand the summand using partial fractions.

5. (Boas 1.6.4) Use the comparison test to prove that series
\[ S_{(a)} := \sum_{n=1}^{\infty} \frac{1}{2^n + 3^n} \quad \text{and} \quad S_{(b)} := \sum_{n=1}^{\infty} \frac{1}{n2^n}. \]

converge.

6. (Boas 1.6.8 and 14) Use the integral test to determine whether the series
\[ S_{(a)} := \sum_{n=1}^{\infty} \frac{n}{n^2 + 4} \quad \text{and} \quad S_{(b)} := \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2 + 9}} \]
converge or diverge.
7. (Boas 1.6.21, 26 and 29) Use the ratio test to determine whether the series

\[ S_{(a)} := \sum_{n=0}^{\infty} \frac{5^n(n!)^2}{(2n)!}, \quad S_{(b)} := \sum_{n=0}^{\infty} \frac{(n!)^3 e^{3n}}{(3n)!} \quad \text{and} \quad S_{(c)} := \sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{(n)!} \]

converge or diverge.

8. (Boas 1.17.1, 4 and 6) Use the alternating series test to determine whether the series

\[ S_{(a)} := \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}, \quad S_{(b)} := \sum_{n=1}^{\infty} \frac{(-3)^n}{n!} \quad \text{and} \quad S_{(c)} := \sum_{n=1}^{\infty} \frac{(-1)^n n}{n + 5} \]

converge or diverge.

9. Which of the series from the previous problem converge absolutely?

10. (Chow A1.13, p. 520) Use Gauss’ test to determine whether the series

\[ S := \left( \frac{1}{2} \right)^2 + \left( \frac{1 \times 3}{2 \times 4} \right)^2 + \left( \frac{1 \times 3 \times 5}{2 \times 4 \times 6} \right)^2 + \cdots \]

converges or diverges. Show that neither the ratio test nor Raabe’s test would be conclusive for this series.