Problem Set VII
Due: Tuesday, 16 November 2010

1. (Problems 14.7.6, 12 and 16, p. 699)
Evaluate the following definite integrals of real functions by relating them to contour integrals in the complex plane.

\[ I^{(a)} := \int_{0}^{2\pi} \frac{d\theta}{(2 + \cos \theta)^2}, \quad I^{(b)} := \int_{0}^{\infty} \frac{x^2 \, dx}{x^4 + 16} \quad \text{and} \quad I^{(c)} := \int_{0}^{\infty} \frac{x \sin x \, dx}{9x^2 + 4}. \]

2. (Problems 14.7.25 and 27, p. 700)
Consider the definite integrals

\[ I^{(a)} := \int_{0}^{\infty} \frac{x \sin x}{9x^2 - \pi^2} \, dx \quad \text{and} \quad I^{(b)} := \int_{0}^{\infty} \frac{\cos \pi x}{1 - 4x^2} \, dx. \]

Determine whether each integral exists. If it does, evaluate it by relating it to a contour integral. If not, then evaluate its Cauchy principal value by relating that integral to a contour integral.

3. (Problems 14.7.34 and 35, p. 700)
Evaluate the definite integrals

\[ I^{(a)} := \int_{0}^{\infty} \frac{\sqrt{x} \, dx}{(1 + x)^2} \quad \text{and} \quad I^{(b)} := \int_{0}^{\infty} \frac{x^{1/3} \, dx}{(1 + x)(2 + x)} \]

by relating each to a contour integral around an appropriate “keyhole contour.”

4. (Problem 14.7.42, p. 701)
Let \( F(z) := f'(z)/f(z) \), where \( f(z) \) is analytic except at isolated points.

a. Show that the residue of \( F(z) \) at an \( n^{th} \)-order zero of \( f(z) \) is \( n \).

   \textbf{Hint:} If \( f(z) \) has a pole of order \( n \) at \( a \), then \( f(z) = a_n (z - a)^n + a_{n+1} (z - a)^{n+1} + \cdots. \)

b. Show that the residue of \( F(z) \) at a \( p^{th} \)-order pole of \( f(z) \) is \( n \).

   \textbf{Hint:} See the definition of a pole of order \( p \) at the end of Section 14.4.