A Two-Domain Spectral Method for Solving the Constraint Equations of Binary Black Holes with Alternative Initial Data Schemes

George Reifenberger

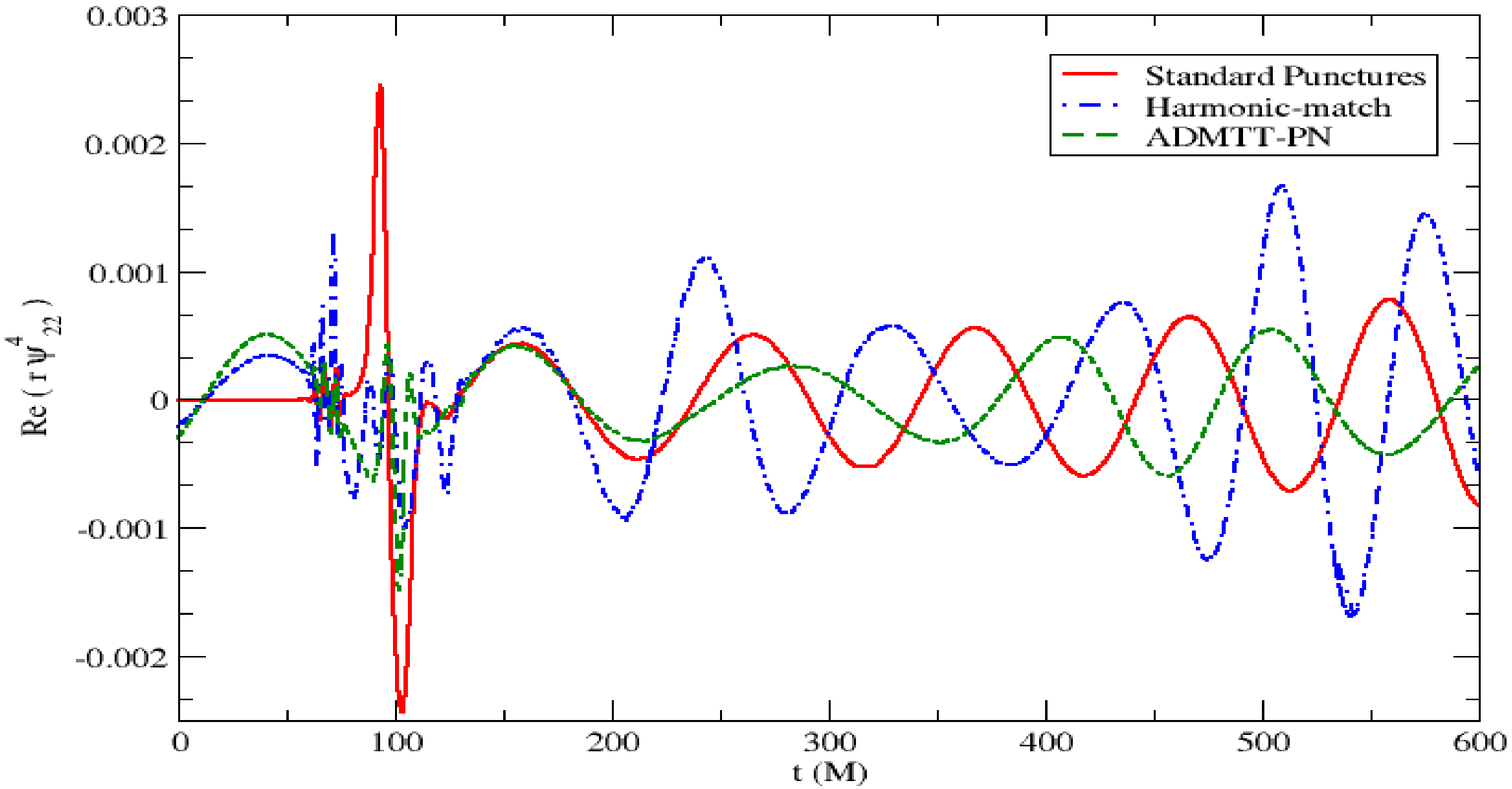
Florida Atlantic University

October 10, 2012

Outline

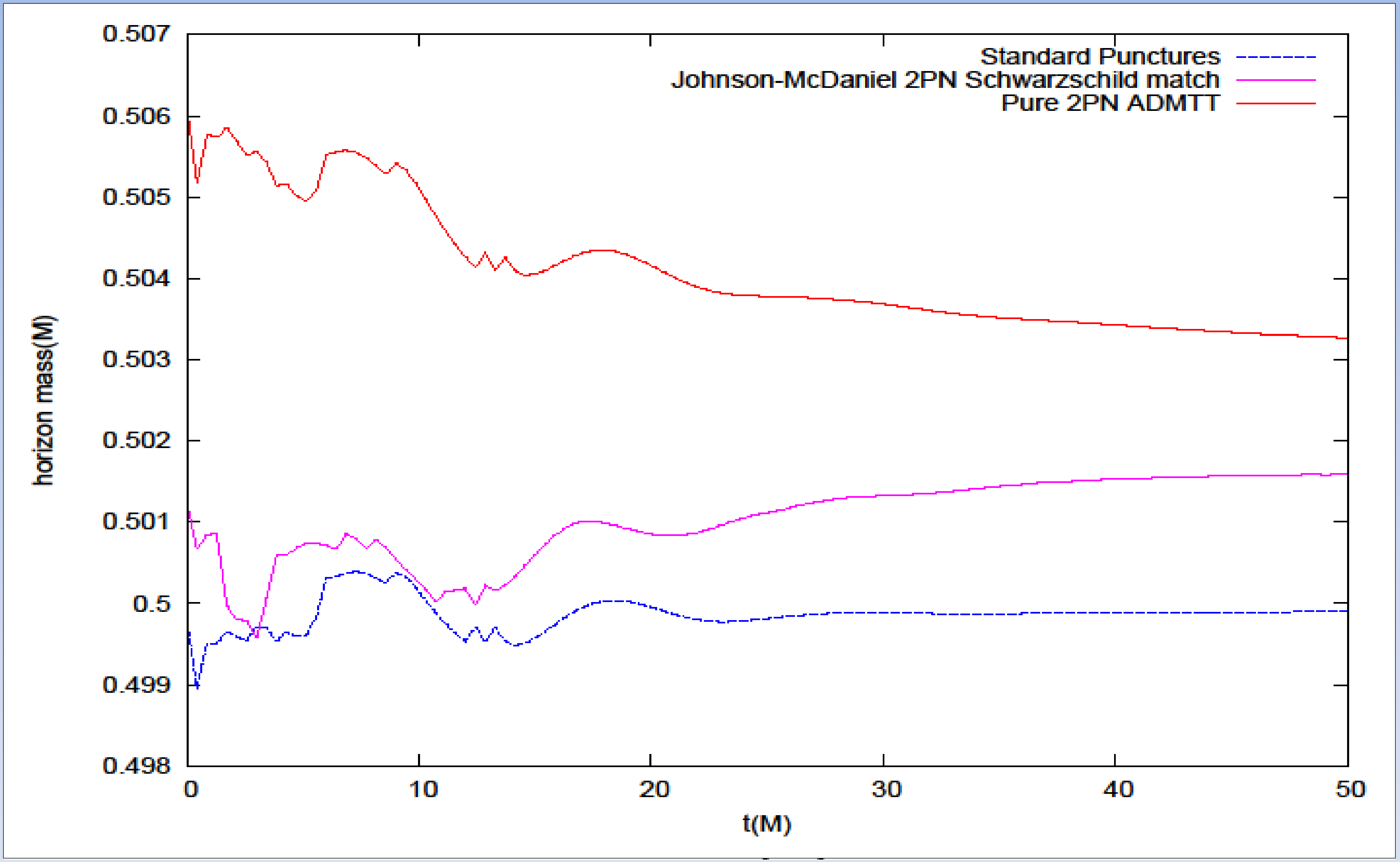
- Motivation
- Two-Domain Grid
- Pseudo-spectral Method
- Constraint Solving

Motivation



Standard Punctures
Harmonic-match
ADMTT-PN
0.003
0.002
0.001
0
-0.001
-0.002
0
100
200
300
400
500
600
t (M)
$Re (r \Psi^{4}_{22})$

Motivation


Standard Punctures
Johnson-McDaniel 2PN Schwarzschild match
Pure 2PN ADMTT
horizon mass(M)
0.507
0.506
0.505
0.504
0.503
0.502
0.501
0.5
0.499
0.498
0
10
20
30
40
50
t(M)

Motivation

- **Constraint solving may further validate schemes**
  - **Use a conformal decomposition and CTTD**

$$g_{ij} = \Psi^4 \, \bar{g}_{ij}$$

$$K^{ij} = \Psi^{-10} \, (\bar{A}^{ij} + \bar{L}W^{ij})$$

  - **Also use the Punctures technique to control divergent behavior**

$$\Psi = \psi + u$$

Motivation

- **Constraint Equations become a set of 4 elliptic equations**

$$\boxed{\begin{gathered} 0 = \overline{\nabla}^2 \Psi - \frac{1}{8} \Psi \overline{R} + \frac{1}{8} \Psi^{-7} (\overline{A}^{ij} + \overline{L}W^{ij})(\overline{A}^{kl} + \overline{L}W^{kl}) \overline{g}_{ik} \overline{g}_{jl} \\ 0 = \overline{\Delta}_L W^i + \overline{\nabla}_j \overline{A}^{ij} \end{gathered}}$$

$$\overline{L}W^{ij} = \overline{\nabla}^i W^j + \overline{\nabla}^j W^i - \frac{2}{3} \overline{g}^{ij} \overline{\nabla}_k W^k \quad , \quad \overline{\Delta}_L W^i = \overline{\nabla}_j \overline{L}W^{ij}$$

Two-Domain Spectral Grid

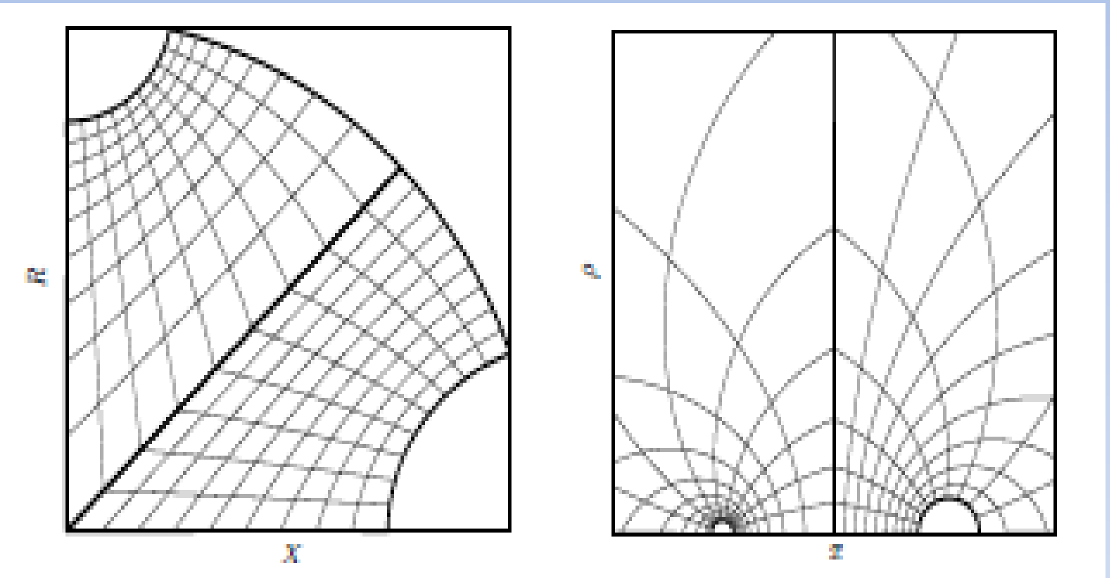
- General method for computing initial data for binaries
  - We choose: Two-Domains for Excised Binary Black Holes

- Developed by Marcus Ansorg as a way to
  - Compactify spatial infinity to a finite grid
  - Resolve data at infinity
  - Deal with puncture singularities (If using Puncture data)

Two-Domain Spectral Grid

- Involves ***several*** coordinate transformations

$$(x, y, z) \rightarrow (\mathrm{x}, \rho, \varphi)$$
$$(x, \rho, \varphi) \rightarrow (\mathrm{X}, \mathrm{R}, \varphi)$$
$$(\mathrm{X}, \mathrm{R}, \varphi) \rightarrow (\mathrm{A}, \mathrm{B}, \varphi)$$



- Grid points are the collocation points of the pseudo-spectral method

Pseudo-spectral Method

- **An alternative to finite-differencing to accurately compute spatial derivatives**
  - **Global vs. Local calculation**
- **A function can be expressed as a set of basis functions**

$$\mathrm{L}\phi = \Phi(\mathrm{x})$$

$$\mathrm{Let}: \phi(\mathrm{x}) = \sum_{\mathrm{m}=0}^{\mathrm{N}} C_m \, \mathrm{f}_{\mathrm{m}}(\mathrm{x})$$

Pseudo-spectral Method

- **E.g. Fourier Series**

$$\text{If}: \quad \nabla_x \phi = \Phi(x) \qquad \text{Let}: \quad \phi(x) = \sum_{m=0}^{N} C_m\, e^{imx}$$

$$\phi_n = \phi(x_n)$$

$$\phi(x_n) = \sum_{m=0}^{N} C_m\, e^{imx_n} \qquad , x_n = \frac{2\pi n}{N}$$

$$\begin{pmatrix} \phi_0 \\ \phi_1 \\ \dots \\ \phi_{n-1} \end{pmatrix} = \begin{pmatrix} & & \\ & M & \\ & & \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ \dots \\ C_{n-1} \end{pmatrix}$$

Pseudo-spectral Method

- **Function values and coefficients stored after first computations**
- **At collocation points we have a 1:1 relation between function values and coefficients through FFTs**
- **Any derivative operation can be handled later with the coefficients (Spectral) or the function values (Pseudo-spectral)**

$$\nabla_x \phi(x_n) = \sum_{m=0}^{N} (im) \cdot C_m \, e^{imx_n} = \sum_{m=0}^{N} (im) \cdot \phi_n$$

Pseudo-spectral Method

- **More efficient as the resolution increases compared to a finite- difference method**

- **Has the potential for “exponential convergence”**

$$\varepsilon \approx \mathrm{O}\left[\left(\frac{1}{\mathrm{N}}\right)^{\mathrm{N}}\right]$$

Pseudo-spectral Method

- Choice of basis functions depends on problem properties
  - *Fourier*, *Chebyshev*, Hermite, Laguerre
- Our domain is suited for Fourier and Chebyshev basis functions
- Our Boundary Conditions determine the collocation points
  - Conformal factor $\longrightarrow$ 0 as r $\longrightarrow$ inf
    - We use the zeroes of the Chebyshev functions as the collocation points

$$\mathrm{A}_l = \frac{1}{2}\left[1 - \mathrm{Cos}\left(\frac{\pi\, l}{\mathrm{n_A} - 1}\right)\right] \qquad \mathrm{B}_l = \frac{1}{2}\left[1 - \mathrm{Cos}\left(\frac{\pi\, l}{\mathrm{n_B} - 1}\right)\right]$$

Constraint Solving

- **Solving multi-dimensional non-linear elliptic equations can be time-consuming and inefficient if done improperly**
- **Standard approach is to use the Newton-Raphson technique**

$$\mathrm{F}_i(x_1, x_2, \ldots, x_N) = 0$$

$$\mathrm{F}_i(\mathbf{x} + \delta\mathbf{x}) = \mathrm{F}_i(\mathbf{x}) + \sum_{j=1}^{N} \frac{\partial \mathrm{F}_i}{\partial x_j} \delta x_j + \mathrm{O}(\delta\mathbf{x}^2)$$

$$\mathbf{F}(\mathbf{x} + \delta\mathbf{x}) = \mathbf{F}(\mathbf{x}) + \mathbf{J} \cdot \delta\mathbf{x} + \mathrm{O}(\delta\mathbf{x}^2)$$

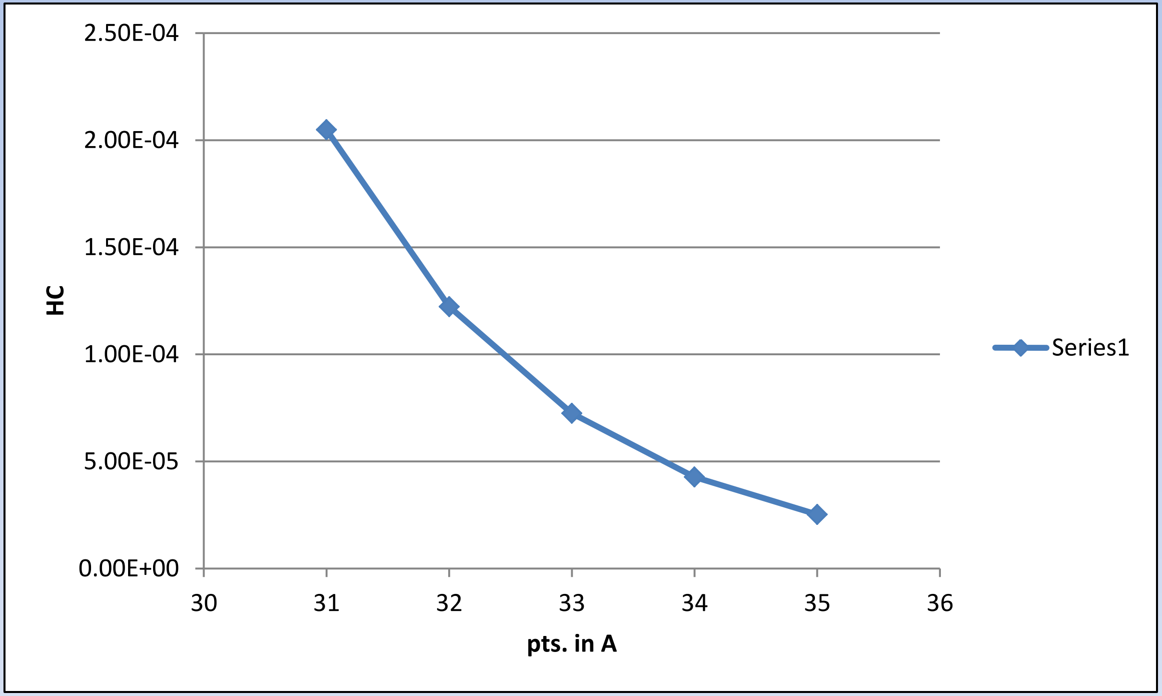
$$\mathbf{J} \cdot \delta\mathbf{x} = -\mathbf{F}$$

Constraint Solving

- Each iteration moves the functions closer to zero until convergence
- Many choices of iteration method (We choose GMRES)
- The speed/efficiency of the N-R routine is highly dependent on the initial guess for each root
- We can reduce the matrix solving time by preconditioning J
  - Same solution with more favorable matrix properties
  - Can use the finite difference J as a preconditioner (More sparse)

Results

- **Convergence for ADMTT near Initial data**



More Results?

- Currently working on solving ADMTT full Initial Data
  - Debugging underway at the moment
- We will also solve the Johnson-McDaniel 2PN Schwarzschild match data
- If our level of accuracy is comparable to punctures data, we will incorporate the solving into our evolutions with BAM
  - Interpolation back onto a finite Cartesian grid of equally-spaced grid points

Bibliography

- Arnowitt, R, Deser S, Misner C.W., *Gravitation: An Introduction to Gravitational Radiation*, edited by L. Witten (Wiley, New York, 1962)
- Jaranowski P, Schafer, G, Third Post-Newtonian higher order ADM Hamilton dynamics for two-body point-mass systems. *Phys. Rev.* D, **57**:7274-7291, 1998
- Wolfgang Tichy, Bernd Brugmann, Manuela Campanelli, and Peter Diener. Binary black hole initial data for numerical general relativity based on Post-Newtonian data. *Phys. Rev.*D, **67**:064008, 2003
- Bernard J. Kelly, Wolfgang Tichy, Bernard F. Whiting. Black hole puncture data with realistic gravitational wave content. *Phys Rev.* D, **76** :024008, 2007
- Kelly B, et. al., Post-Newtonian Initial Data with Waves: Progress in Evolution, arXiv:0912.5311v1 [gr-qc]
- Baumgarte T, Shapiro S, Numerical Relativity and Compact Binaries, *Phys. Rept.* **376**:41-131, 2003
- Marcus Ansorg, Multi-Domain Spectral Method for Initial Data of Arbitrary Binaries in General Relativity. *Class. Quantum Grav.* **24** S1
- Wolfgang Tichy, A new numerical method to construct binary neutron star initial data. *Class. Quantum Grav.* **26** 175018