A Two-Domain Spectral Method for Solving the Constraint Equations of Binary Black Holes with Alternative Initial Data Schemes

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October 10, 2012
Outline

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Motivation
Motivation

This is only a first step because we have not strongly enforced the constraint equations (as with puncture data).
Motivation

• Constraint solving may further validate schemes
  – Use a conformal decomposition and CTTD
    \[ g_{ij} = \Psi^4 \bar{g}_{ij} \]
    \[ K^{ij} = \Psi^{-10} (\bar{A}^{ij} + \bar{L}W^{ij}) \]
  – Also use the Punctures technique to control divergent behavior
    \[ \Psi = \psi + u \]
Motivation

- Constraint Equations become a set of 4 elliptic equations

\[
0 = \bar{\nabla}^2 \psi - \frac{1}{8} \psi R + \frac{1}{8} \psi^{-7} (\bar{A}^{ij} + \bar{L} W^{ij}) (\bar{A}^{kl} + \bar{L} W^{kl}) g_{ik} g_{jl} \\
0 = \bar{\Delta}_L W^i + \bar{\nabla}_j \bar{A}^{ij}
\]

\[
\bar{L} W^{ij} = \bar{\nabla}^i W^j + \bar{\nabla}^j W^i - \frac{2}{3} g^{ij} \bar{\nabla}^k W^k , \quad \bar{\Delta}_L W^i = \bar{\nabla}_j \bar{L} W^{ij}
\]
Two-Domain Spectral Grid

• General method for computing initial data for binaries
  — We choose: Two-Domains for Excised Binary Black Holes

• Developed by Marcus Ansorg as a way to
  — Compactify spatial infinity to a finite grid
  — Resolve data at infinity
  — Deal with puncture singularities (If using Puncture data)
Two-Domain Spectral Grid

• Involves several coordinate transformations

\[(x, y, z) \rightarrow (x, \rho, \varphi)\]
\[(x, \rho, \varphi) \rightarrow (X, R, \varphi)\]
\[(X, R, \varphi) \rightarrow (A, B, \varphi)\]

• Grid points are the collocation points of the pseudo-spectral method
Pseudo-spectral Method

• An alternative to finite-differencing to accurately compute spatial derivatives
  — Global vs. Local calculation

• A function can be expressed as a set of basis functions

\[ L\phi = \Phi(x) \]

Let: \[ \phi(x) = \sum_{m=0}^{N} C_m f_m(x) \]
Pseudo-spectral Method

- E.g. Fourier Series

If: \( \nabla_x \phi = \Phi(x) \)

Let: \( \phi(x) = \sum_{m=0}^{N} C_m e^{imx} \)

\( \phi_n = \phi(x_n) \)

\( \phi(x_n) = \sum_{m=0}^{N} C_m e^{imx_n} \), \( x_n = \frac{2\pi n}{N} \)
Pseudo-spectral Method

• Function values and coefficients stored after first computations

• At collocation points we have a 1:1 relation between function values and coefficients through FFTs

• Any derivative operation can be handled later with the coefficients (Spectral) or the function values (Pseudo-spectral)

\[ \nabla_x \phi(x_n) = \sum_{m=0}^{N} (im) \cdot C_m \cdot e^{imx_n} = \sum_{m=0}^{N} (im) \cdot \phi_n \]
Pseudo-spectral Method

• More efficient as the resolution increases compared to a finite-difference method

• Has the potential for “exponential convergence”

\[ \varepsilon \approx O \left( \left( \frac{1}{N} \right)^N \right) \]
Pseudo-spectral Method

• Choice of basis functions depends on problem properties
  — Fourier, Chebyshev, Hermite, Laguerre

• Our domain is suited for Fourier and Chebyshev basis functions

• Our Boundary Conditions determine the collocation points
  — Conformal factor $\to 0$ as $r \to \infty$
    • We use the zeroes of the Chebyshev functions as the collocation points

\[
A_i = \frac{1}{2} \left[ 1 - \cos \left( \frac{\pi l}{n_A - 1} \right) \right] \quad B_i = \frac{1}{2} \left[ 1 - \cos \left( \frac{\pi l}{n_B - 1} \right) \right]
\]
Constraint Solving

- Solving multi-dimensional non-linear elliptic equations can be time-consuming and inefficient if done improperly.

- Standard approach is to use the Newton-Raphson technique:

\[
F_i(x_1, x_2, \ldots, x_N) = 0
\]

\[
F_i(x + \delta x) = F_i(x) + \sum_{j=1}^{N} \frac{\partial F_i}{\partial x_j} \delta x_j + O(\delta x^2)
\]

\[
F(x + \delta x) = F(x) + J \cdot \delta x + O(\delta x^2)
\]

\[
J \cdot \delta x = -F
\]
Constraint Solving

• Each iteration moves the functions closer to zero until convergence

• Many choices of iteration method (We choose GMRES)

• The speed/efficiency of the N-R routine is highly dependent on the initial guess for each root

• We can reduce the matrix solving time by preconditioning J
  — Same solution with more favorable matrix properties
  — Can use the finite difference J as a preconditioner (More sparse)
Results

• Convergence for ADMTT near Initial data
More Results?

• Currently working on solving ADMTT full Initial Data
  — Debugging underway at the moment

• We will also solve the Johnson-McDaniel 2PN Schwarzschild match data

• If our level of accuracy is comparable to punctures data, we will incorporate the solving into our evolutions with BAM
  — Interpolation back onto a finite Cartesian grid of equally-spaced grid points
Bibliography


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