Path integral measure as determined by canonical gravity

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FAUST Seminar Spring 2013
Motivation

• **Dynamics** of current spin foam approach is independent from canonical theory

• Need to use **Liouville measure** on the reduced phase space which preserves the phase space volume

\[ \int \mathcal{D}q \exp iS \rightarrow \int \mathcal{D}q\mathcal{D}p \exp iS \]

• **Plebanski-Holst formulation:**

  has desired variables \( (\omega_{IJ}^{\mu}, X_{\mu}^{IJ}) \) \( \rightarrow \) need to integrate out the connection
Outline

• Reduced phase space path integral approach for a general Hamiltonian system with constraints
• Applying that to Plebanski Holst formulation
• Integrating out the connection
• ADM path integral
• Conclusion
Reduced phase space path integral

(Quantization of Gauge Systems, Henneaux and Teitelboim)

- First class constraints: \[ \{F_i, F_j\} = f_{ij}^k F_k \approx 0 \]
- Second class constraints: \[ \{S_i, S_j\} \neq 0 \]

Liouville measure on the whole phase space: \( dqdp \)

Liouville measure on the reduced phase space: \( dq^* dp^* \)

Liouville measure for noncanonical coordinates in phase space: \[ [\det\{x^\mu, x^\nu\}]^{-\frac{1}{2}} dx^\mu \]

- Change of variables: \((q, p) \rightarrow (q^*, p^*, S)\)

\[ dq^* dp^* dS [\det\{S, S\}]^{-\frac{1}{2}} = dqdp \quad \rightarrow \quad \int dq^* dp^* = \int \delta(S) [\det\{S, S\}]^{\frac{1}{2}} dqdp \]
Reduced phase space path integral

(Quantization of Gauge Systems, Henneaux and Teitelboim)

Gauge fixing: \((F, \xi) \rightarrow\) second class

\[
\int dq^* dp^* = \int \delta(S)[\det\{S, S\}]^{\frac{1}{2}} dq dp
\]

\[
\{\{F, F\}, \{F, \xi\}, \{\xi, F\}, \{\xi, \xi\}\} \rightarrow [\det\{F, \xi\}]^2
\]

Faddeev Popov term:

\[
[\det\{S, S\}]^{\frac{1}{2}} \rightarrow \Delta_{FP} = |\det\{F, \xi\}|
\]

\[
\delta(S) \rightarrow \delta(F)\delta(\xi)
\]

Path integral:

\[
\mathcal{Z} := \int \mathcal{D}q\mathcal{D}p \exp iS
\]

\[
\mathcal{Z} := \int \mathcal{D}q\mathcal{D}p \sqrt{\det\{\{S, S\}\}}|\det\{F, \xi\}|\delta[S]\delta[F]\delta[\xi] \exp iS
\]

Expectation value:

\[
\mathcal{Z}(\mathcal{O}) := \int \mathcal{D}q\mathcal{D}p \sqrt{\det\{\{S, S\}\}}|\det\{F, \xi\}|\mathcal{O}\delta[S]\delta[F]\delta[\xi] \exp iS
\]
Plebanski Holst formulation

• P-H action:

\[ S_{PH} = \int X_{IJ} \wedge F^{IJ} \]

• Conjugate variables: \((\omega^{IJ}_{\mu\nu}, X^{IJ}_{\mu\nu})\)

• Simplicity constraint:

\[ C_{\mu\nu\rho\sigma} := \epsilon_{IJKL} X^{IJ}_{\mu\nu} X^{KL}_{\rho\sigma} - \frac{s}{4!} \epsilon_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta} \epsilon_{IJKL} X^{IJ}_{\alpha\beta} X^{KL}_{\gamma\delta} \approx 0 \]

\[ X^{IJ} = \begin{cases} \pm \frac{1}{e} e^I \wedge e^J & (I\pm) \\ \pm \frac{1}{2\kappa} e^I_{KL} e^K \wedge e^L & (II\pm) \end{cases} \]

• Reduced phase space integral\(^{(1)}\):

\[ Z = \int_{(II+)} D\omega^{IJ}_{\mu} D X^{IJ}_{\mu\nu} \delta^{20}(C) \nu^9 V_s \exp i \int X_{IJ} \wedge F^{IJ} \]

\[ \nu_s = h^{\frac{1}{2}} \]

\[ \nu = g^{\frac{1}{2}} \]

\(^{(1)}\) Canonical path integral measures for Holst and Plebanski gravity: I. Reduced phase space derivation. J. Engle, M. Han and T. Thiemann, 2010 Class. Quantum Grav. 27 245014
Integrating out the connection

\[ S[X, \omega] = (\omega, \hat{A}\omega) + (b, \omega) \]

\[ (\alpha, \beta) := \int d^4x \delta^{\mu\nu} \alpha_\mu^{IJ} \beta_\nu^{IJ} \]

Gaussian integral:

\[ I(X) = \int D\omega \exp i [(\omega, \hat{A}\omega) + (b, \omega)] \]

\[ I(X) \overset{\triangle}{=} (\det \hat{A})^{-\frac{1}{2}} \exp \frac{-i}{4} (b, \hat{A}^{-1}b) \]

Determinant of \( A \):

\[ \det \hat{A} = \mathcal{V}^{12} \]

\[ Z = \int_{(II+)} D\omega^{IJ}_\mu D X^{IJ}_\mu \delta^{20}(C) \mathcal{V}^{9} V_s \exp iS \]

\[ Z^{(1)} \overset{\triangle}{=} \int D X^{IJ}_\mu \delta^{20}(C) \mathcal{V}^{3} V_s \exp iS \]

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1) Pure momentum path integral for spin foam, A. Ch. Shirazi & J. Engle (In Preparation)
ADM Formulation

- Canonical variables: \((h_{ab}, \pi^{ab})\)
  \[\pi^{ab} = \frac{\partial L}{\partial \dot{h}_{ab}}\]

- First class constraints: Hamiltonian \(H_0\) and 3-diff constraints \(H_a\)
  \[
  H_0 = -\frac{1}{2} h^{-1/2} [h_{ac} h_{bd} + h_{ad} h_{bc} - h_{ab} h_{cd}] \pi^{ab} \pi^{cd} - (3) R(h) h^{1/2}
  
  H_a = h^{-1/2} h_{ab} D_c \pi^{bc}
  
  - No second class constraints

\[
Z := \int \mathcal{D}q \mathcal{D}p \sqrt{\text{det} \{S, S\}} \Delta_{FP} \delta[S] \delta[F] \delta[\xi] \exp iS
\]

\[
Z_{ADM} = \int \mathcal{D}h_{ab} \mathcal{D}\pi^{ab} \mathcal{D}N \mathcal{D}N^a \exp i \int d^4x (\pi^{ab} \dot{h}_{ab} - \mathcal{H}_G(h_{ab}, \pi^{ab}))
\]

Using:
\[
\int \mathcal{D}N \exp -iNH = \delta(H), \quad \int \mathcal{D}N^a \exp -iN^aH_a = \delta(H_a)
\]

- \(N\) and \(N^a\) are Lagrange multipliers
- No gauge fixing: No Faddeev-popov term
Final ADM path integral

\[ Z_{ADM} = \int D h_{ab} D \pi^{ab} D N D N^a \exp i \int dt (\hat{A}_{ab,cd} \pi^{ab} \pi^{cd} + \hat{B}_{ab} \pi^{ab} + \hat{C}) \]

\[ \hat{A}_{ab,cd} := -\frac{N h^{-\frac{1}{2}}}{2} [h_{ac} h_{bd} + h_{ad} h_{bc} - h_{ab} h_{cd}] \]

\[ \hat{B}_{ab} := \dot{h}_{ab} - 2 D_{(a} N_{b)} \]

\[ \hat{C} := N h^{\frac{1}{2}}(3) R \]

- Integrating out \( \pi^{ab} \)

\[ Z_{ADM} = \int (\det \hat{A})^{-\frac{1}{2}} D h_{ab} D N D N^a \exp i \int d^4x (\frac{-1}{4} \hat{B}_{ab} \hat{A}^{-1ab,cd} \hat{B}_{cd} + \hat{C}) \]

The exponential: \( (\frac{-1}{4} \hat{B}_{ab} \hat{A}^{-1ab,cd} \hat{B}_{cd} + \hat{C}) \to \mathcal{L}_G \) \quad Determinant: \( (\det \hat{A})^{-\frac{1}{2}} = N^{-3} h^{-\frac{1}{2}} \)

- Final ADM path integral\(^{(1)}\)

\[ Z_{ADM} = \int N^{-3} h^{-\frac{1}{2}} D h_{ab} D N D N^a \exp i \int d^4x \mathcal{L}_G. \]

\(^{(1)}\) Pure momentum path integral for spin foam, A. Ch. Shirazi & J. Engle (In Preparation)
Comparing to our results

(1) \[ Z_{ADM} \overset{\sim}{=} \int N^{-3} h^{-1/2} \mathcal{D}h_{ab} \mathcal{D}NdN^a \exp iS_G \]

(2) \[ Z \overset{\sim}{=} \int \mathcal{D}X^{IJ}_{\mu \nu} \delta^{20}(C) \sqrt{V}V_s \exp iS_G \]

- **Change of variables** \((h_{ab}, N, N^a) \rightarrow (g_{\mu \nu})^{(1)}\)

\[ \mathcal{D}g_{\mu \nu} = \mathcal{D}g_{ab} \mathcal{D}g_{00} \mathcal{D}g_{0a} = \det J \mathcal{D}h_{ab} \mathcal{D}N \mathcal{D}N^a \]

\[ g_{ab} = h_{ab} \]
\[ g_{00} = -N^2 + h_{ab}N^aN^b \]
\[ g_{0a} = h_{ab}N^b \]

\[ \det J = -2hN \]
\[ \mathcal{D}g_{\mu \nu} = -2hN \mathcal{D}h_{ab} \mathcal{D}N \mathcal{D}N^a \]

\[ \frac{\partial (g_{ab}, g_{00}, g_{0a})}{\partial (h_{ab}, N, N^a)} = \begin{vmatrix} N & 0 & -2N & 0 \\ N^a & 0 & 2N^a & h_{ab} \\ h_{ab} & 1 & 0 & N^a \end{vmatrix} \]

\[ Z_{ADM} \overset{\sim}{=} \int N^{-4} h^{-3/2} \mathcal{D}g_{\mu \nu} \exp iS_G \]

- **Other change of variables:**

\[ X^{IJ}_{\mu \nu} \rightarrow (e^I_{\mu}, C)^{(2)} \quad \mathcal{D}X^{IJ}_{\mu \nu} = \sqrt{V}e^I_{\mu} \mathcal{D}^{20}C \]

\[ (e^I_{\mu}) \rightarrow (g_{\mu \nu}, \Lambda^J_I)^{(3)} \quad \mathcal{D}g_{\mu \nu} \mathcal{D}\Lambda^J_I = \sqrt{g}e^I_{\mu} \]

1) Pure momentum path integral for spin foam, A. Ch. Shirazi & J. Engle (In Preparation)
2) Canonical path integral measures for Holst and Plebanski gravity: I. Reduced phase space derivation, J. Engle, M. Han and T. Thiemann, 2010 Class. Quantum Grav. 27 245014
3) Path integral measure for first-order and metric gravities, R. Aros, M. Contreras and J. Zanelli, 2003 Class. Quantum Grav. 20 2937
Conclusion

- Breaking the manifest general covariance because of the **appearance of 3-volume**.
  
  **Reason**: Foliation of space-time! Price to pay for consistency with the canonical theory.

- The path integral is invariant under **Bergman-Komar “group”** but not Diff(M)!
  
  (Canonical Path-Integral Measures for Holst and Plebanski Gravity. II. Gauge Invariance and Physical Inner Product. M. Han, Class. Quant. Grav. 27 (2010) 245015)

- **Continuum path integral**

  **Next step**: Discretizing and quantizing the measure and implying to spin foam models.