Final Exam

1 a) We begin by analyzing the problem in the instantaneously co-moving inertial frame. There, conservation of energy and momentum give

$$d\mathring{E} + \gamma(u)\,d\mu = 0 \qquad \text{and} \qquad d\mathring{p} + \gamma(u)\,d\mu \cdot -u = 0$$

Here, $d\mu$ is the rest-mass of the material ejected by the rocket in an infinitesimal interval $d\tau$ of proper time. The excess loss of energy in the rocket due to the factor $\gamma(u)$ may be attributed to the binding energy used to eject $d\mu$ at speed u.

The fixed frame moves with velocity $-v$ relative to the instantaneously co-moving one, so the Lorentz transformation gives

$$dE = \gamma(v)\left[d\mathring{E} + v\,d\mathring{p}\right] \qquad dp = \gamma(v)\left[d\mathring{p} + v\,d\mathring{E}\right]$$

$$= \gamma(v)\gamma(u)\,d\mu\left[uv - 1\right] \qquad = \gamma(v)\gamma(u)\,d\mu\left[u - v\right]$$

But we have

$$dp = d(m\gamma(v)v) = d(m\gamma(v))\,v + m\gamma(v)\,dv$$

$$\Rightarrow \quad m\gamma(v)\,dv = dp - v\,dE = \gamma(v)\gamma(u)\,d\mu\left[u - v^2u\right]$$

$$= \gamma^{-1}(v)\gamma(u)\,u\,d\mu$$

2

But $\mathring{E} = m$ in the co-moving frame, so

$$dm = d\mathring{E} = -\gamma(u)\,d\mu$$

$$\Rightarrow m\gamma(v)\,dv = -\gamma^{-1}(v)\,u\,dm$$

$$\Rightarrow \frac{dv}{1-v^2} = -u\,\frac{dm}{m}$$

$$\Rightarrow \frac{1}{2}\ln\frac{1+v}{1-v} = -u\ln\frac{m}{m_0}$$

$$\Rightarrow m = m_0\left(\frac{1-v}{1+v}\right)^{1/2u}$$

This is the result.

b) In non-relativistic theory, we again use the instantaneously co-moving frame to write

$$m\,d\mathring{v} - (-dm)\,u = 0 \quad \Rightarrow \quad \frac{dm}{m} = -\frac{d\mathring{v}}{u} = -\frac{dv}{u}$$

$$\Rightarrow \ln\frac{m}{m_0} = -\frac{v}{u} \quad \Rightarrow \quad m = m_0 e^{-v/u}$$

Here, the differential of the velocity is the same in the fixed and co-moving frames.

To show equivalence in the non-relativistic limit, we recall that

$$\lim_{s\to\infty}\left(1+\frac{z}{s}\right)^s = e^z$$

Defining $s = \frac{1}{u}$ and $z = \frac{v}{u}$, we have

$$m = m_0\sqrt{\frac{(1-z/s)^s}{(1+z/s)^s}} \to m_0\sqrt{\frac{e^{-z}}{e^{z}}} = m_0 e^{-z} = m_0 e^{-v/u}$$

c) The difference between the relativistic and non-relativistic rocket problems, apart from the use of Lorentz vs. Galilei transformations, is that energy is conserved in the former case, mass in the latter. Physically, binding energy of the fuel in relativistic theory has mass, unlike in non-relativistic physics.

2 a) Tom will follow a time-like curve in the interior Schwarzschild geometry

$$ds^2 = \left(\frac{2M}{r} - 1\right) dt^2 - \left(\frac{2M}{r} - 1\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Noting that $ds^2 = -d\tau^2$ along a time-like curve, we have

$$-1 = -\left(\frac{2M}{r} - 1\right)^{-1} \dot{r}^2 + \left(\frac{2M}{r} - 1\right) \dot{t}^2 + r^2\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right)$$

$$\Rightarrow \dot{r}^2 = \left(\frac{2M}{r} - 1\right) + \left(\frac{2M}{r} - 1\right)^2 \dot{t}^2 + \left(\frac{2M}{r} - 1\right) r^2\left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2\right)$$

All terms on the right are positive, so

$$\dot{r}^2 \geq \frac{2M}{r} - 1 \qquad \Rightarrow \qquad -\dot{r} \geq \sqrt{\frac{2M}{r} - 1}$$

Here, we have noted that all future-directed time-like curves in the interior have $\dot{r} < 0$. Thus, we find

$$d\tau \leq -\left(\frac{2M}{r} - 1\right)^{-1/2} dr \qquad \Rightarrow \qquad \tau \leq \int_{r_0}^{r_1} \frac{-dr}{\sqrt{\frac{2M}{r} - 1}}$$

This bounds the proper time along time-like curves from radius r_0 to radius r_1. Setting $r_0 = 2M$ and $r_1 = 0$, we have

$$\tau \leq \int_0^{2M} \frac{dr}{\sqrt{\frac{2M}{r} - 1}} \qquad r = 2M\cos^2\theta$$

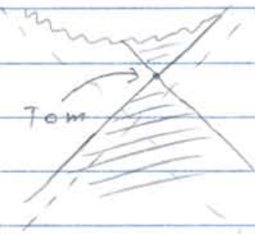
$$dr = -4M\cos\theta\sin\theta\, d\theta$$

$$= \int_{\frac{\pi}{2}}^{0} \frac{-4M\cos\theta\sin\theta\, d\theta}{\tan\theta}$$

$$= 4M\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 2M\int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta = \pi M$$

Now, what path should Tom follow to maximize the proper time along the curve? If we knew his initial and final positions in space time, then the answer would be that he should not fire his thrusters at all and follow a geodesic. But we don't know his final position, we know his initial position and initial velocity. It is difficult to imagine how to set up a variational problem in which surface terms arise only at the initial time under variation, which then could be set to zero using these boundary conditions. So, the problem is difficult to set up mathematically. Moreover, the inequality above is saturated in the case of radial infall from rest at the horizon, so maybe Tom's best move is to accelerate rapidly to put himself on such a radial trajectory and then just enjoy the ride.



b) These questions can be answered using the Penrose diagram. Tom cannot see the singularity because it is in the future. Therefore, he does not know he has passed through a horizon. He cannot radio Ground Control with his shirt preference because all of his radio signals end up on the singularity at $r=0$.

c) He can, however, hear radio signals coming in to the horizon. He can hear us.

d) Yes.

3 a) We will use the Cartan method on the orthonormal dual basis

$$e^\alpha = \begin{pmatrix} dt \\ X\,dx \\ Y\,dy \\ dz \end{pmatrix} \rightsquigarrow \eta_{\alpha\beta}\, de^\beta = \begin{pmatrix} 0 \\ X'\,du\wedge dx \\ Y'\,du \wedge dy \\ 0 \end{pmatrix}$$

Recalling that $du = dt - dz$, we can read off

$$-\omega_{\alpha\beta} = \begin{pmatrix} 0 & X'dx & Y'dy & 0 \\ -X'dx & 0 & 0 & X'dx \\ -Y'dy & 0 & 0 & Y'dy \\ 0 & -X'dx & -Y'dy & 0 \end{pmatrix}$$

One can check that this is the full matrix

We rewrite this result as

$$\omega_\alpha{}^\beta = \begin{pmatrix} 0 & -X'dx & -Y'dy & 0 \\ -X'dx & 0 & 0 & -X'dx \\ -Y'dy & 0 & 0 & -Y'dy \\ 0 & X'dx & Y'dy & 0 \end{pmatrix}$$

One cane easily check that $\omega\wedge\omega = 0$, so

$$R_\alpha{}^\beta = d\omega_\alpha{}^\beta = \begin{pmatrix} 0 & -X''du\wedge dx & -Y''du\wedge dy & 0 \\ -X''du\wedge dx & 0 & 0 & -X''du\wedge dx \\ -Y''du\wedge dy & 0 & 0 & -Y''du\wedge dy \\ 0 & X''du\wedge dx & Y''du\wedge dy & 0 \end{pmatrix}$$

The corresponding vector basis is

$$e_\alpha = \begin{pmatrix} \partial_t \\ X^{-1}\partial_x \\ Y^{-1}\partial_y \\ \partial_z \end{pmatrix} \Rightarrow R_\alpha = R_\alpha{}^\beta \llcorner e_\beta = \begin{pmatrix} -\left(\frac{X''}{X} + \frac{Y''}{Y}\right) du \\ 0 \\ 0 \\ \left(\frac{X''}{X} + \frac{Y''}{Y}\right) du \end{pmatrix}$$

The vacuum Einstein equations are equivalent to $R_{ab} = 0$, so the result follows immediately.

b) The Killing fields obviously include ∂_x and ∂_y, which define the translational symmetries of the planar wave-fronts. Slightly less obvious is $\partial_t + \partial_z$, which corresponds to the fact that the surfaces of constant u are phase fronts of this non-linear wave in spacetime. There are no other symmetries.

4 a) We derived in class the geodesic equation

$$\frac{d\phi}{dU} = \left[b^{-2} - U^2 + 2MU^3\right]^{-1/2} \quad \text{with } U := \frac{1}{r}$$

in the case $K=0$. The point of closest approach has $\frac{dr}{d\phi} = 0$, so that

$$\left(b^{-2} - U^2 + RU^3\right)^{1/2} = 0 \quad \Rightarrow \quad \frac{r^3}{b^2} - r + R = 0$$

We multiply this result by $\frac{\sqrt{3}^3}{2b}$ to find

$$4\left(\frac{\sqrt{3}\,r}{2b}\right)^3 - 3\left(\frac{\sqrt{3}\,r}{2b}\right) + \frac{\sqrt{3}^3 R}{2b} = 0$$

The largest positive real root of this cubic gives the turning point. Using the hint, we set

$$\frac{\sqrt{3}\,r_\pm}{2b} = \pm\cos\theta \quad \Rightarrow \quad 4\cos^3\theta - 3\cos\theta \pm \frac{\sqrt{3}^3 R}{2b} = 0$$

We can solve this when

$$\pm\frac{\sqrt{3}^3 R}{2b} = -\cos 3\theta \quad \Rightarrow \quad \theta = \frac{1}{3}\cos^{-1}\left(\mp 3\,\frac{\sqrt{3}\,R}{2b}\right)$$

$$\Rightarrow \frac{\sqrt{3}\,r_\pm}{2b} = \pm\cos\left(\frac{1}{3}\cos^{-1}\left(\mp 3\,\frac{\sqrt{3}\,R}{2b}\right)\right)$$

To find the third root, we set

$$\frac{\sqrt{3}\,r_0}{2b} = \sin\theta \quad \Rightarrow \quad 4\sin^3\theta - 3\sin\theta + \frac{\sqrt{3}^3 R}{2b} = 0$$

$$\Rightarrow \frac{\sqrt{3}^3 R}{2b} = \sin 3\theta \quad \Rightarrow \quad \theta = \frac{1}{3}\sin^{-1}\left(3\,\frac{\sqrt{3}\,R}{2b}\right)$$

$$\Rightarrow \frac{\sqrt{3}\,r_0}{2b} = \sin\left(\frac{1}{3}\sin^{-1}\left(3\,\frac{\sqrt{3}\,R}{2b}\right)\right)$$

We can check which of these is physical by taking the limit $b \to \infty$ in which the light ray should be undeflected. Keeping terms to order b^{-1},

$$\pm \cos\left(\frac{1}{3}\cos^{-1}\left(\mp 3\frac{\sqrt{3}R}{2b}\right)\right) = \pm \cos\left(\frac{1}{3}\left(\frac{\pi}{2} \pm 3\frac{\sqrt{3}R}{2b}\right)\right)$$

$$\cong \pm\left[\cos\frac{\pi}{6} - \sin\frac{\pi}{6}\cdot \pm\frac{\sqrt{3}R}{2b}\right]$$

$$= \pm\frac{\sqrt{3}}{2} - \frac{\sqrt{3}R}{4b} \quad \Rightarrow \quad r_\pm \cong \pm b - \frac{1}{2}R$$

$$\sin\left(\frac{1}{3}\sin^{-1}\left(3\frac{\sqrt{3}R}{2b}\right)\right) \cong \sin\left(\frac{\sqrt{3}R}{2b}\right) \cong \frac{\sqrt{3}R}{2b}$$

$$\Rightarrow \quad r_0 \cong R$$

Clearly, the upper sign in the cosine expression is physical, which is the result.

b) The discriminant

$$\Delta = \left(\frac{1}{3}\cdot -b^2\right)^3 + \left(-\frac{1}{2}\cdot R b^2\right)^2 = \frac{b^4}{108}\left(27R^2 - 4b^2\right)$$

becomes positive if $b^2 < \frac{27}{4}R^2$, which signals a single real root for the cubic. Using the hint, we can write

$$\frac{\sqrt{3}r_-}{2b} = -\cosh u \quad \Rightarrow \quad 4\cosh^3 u - 3\cosh u - \frac{\sqrt{3}^3 R}{2b} = 0$$

$$\Rightarrow \frac{\sqrt{3}^3 R}{2b} = \cosh 3u \quad \Rightarrow \quad u = \frac{1}{3}\cosh^{-1}\left(3\frac{\sqrt{3}R}{2b}\right)$$

$$\Rightarrow \frac{\sqrt{3}r_-}{2b} = -\cosh\left(\frac{1}{3}\cosh^{-1}\left(3\frac{\sqrt{3}R}{2b}\right)\right) < 0$$

c) The key observation here is that as $b \to \frac{\sqrt{3}\,3R}{2}$ from above, the physical turning radius obeys

$$\frac{\sqrt{3}\,r_+}{2b} \to \cos\left(\frac{1}{3}\cos^{-1}\left(-3\,\frac{\sqrt{3}\,R}{2b}\right)\right)$$

$$= \cos\left(\frac{1}{3}\cos^{-1}(-1)\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\Rightarrow \quad r_+ \to \frac{b}{\sqrt{3}} = \frac{3}{2}R = 3M$$

This is the radius of the unstable circular null geodesic found in class. When b is slightly larger than this critical value, the orbit will spiral around just outside this circle many times before turning and going back out.

d) The integral is given by

$$\Delta\phi = 2\int_0^{u_+} \frac{du}{(b^{-2} - u^2 + Ru^3)^{1/2}}$$

Now, we can write the cubic in this integral as

$$b^{-2} - u^2 + Ru^3 = b^{-2} r^{-3} (r^3 - b^2 r + Rb^2)$$

$$= b^{-2} r^{-3} (r - r_+)(r - r_0)(r - r_-)$$

$$= b^{-2} r_+ r_0 r_- (u_+ - u)(u_0 - u)(u_- - u)$$

From the second equality, it follows that $-r_+ r_0 r_- = Rb^2$, whence

$$b^{-2} - u^2 + Ru^3 = -R(u_+ - u)(u_0 - u)(u_- - u)$$

Since only u_- is negative, we may write

$$\Delta\phi = 2\int_0^{u_+} \frac{du}{\sqrt{(u_+-u)(u_0-u)}\sqrt{R(u.-u_-)}}$$

Now, when $b^2 > \frac{27}{4}R^2$, the roots u_+ and u_0 are distinct with $u_+ > u_0 > 0$. But in the critical limit they degenerate. The integrand always diverges at $u = u_+$, but typically only like $x^{-1/2}$, which integrates to $x^{1/2}$. In the critical limit, however, $u_0 = u_+$, and the integrand diverges like x^{-1}. The integral therefore diverges logarithmically at its upper limit. Physically, this merely reflects the many cycles the photon will execute around the unstable circular orbit in that limit.

In classical physics, the absorption cross-section is merely the area of the disk of photons that disappear down the hole:

$$A = \pi\left(\frac{27}{4}R^2\right) = 27 M^2.$$

The end.