Problem Set V

1 To calculate κ_{ab}, we must find $\nabla_a \hat{r}_b$. But on Euclidean space we have

$$g_{ab} = \nabla_a r_b = \nabla_a (r \hat{r}_b) = r \nabla_a \hat{r}_b + \hat{r}_b \nabla_a r$$

$$\Rightarrow \nabla_a \hat{r}_b = \frac{1}{r}(g_{ab} - \hat{r}_a \hat{r}_b)$$

The last result follows from

$$2r\,dr = dr^2 = d(x^2+y^2+z^2) = 2x\,dx + 2y\,dy + 2z\,dz$$

$$\Rightarrow \nabla_a r = \frac{x}{r} e^x_a + \frac{y}{r} e^y_a + \frac{z}{r} e^z_a = \frac{r_a}{r} = \hat{r}_a$$

Thus, $\nabla_a \hat{r}_b$ is already projected on both indices, and the result follows immediately.

2 a) The vectors $\hat{U}_a$ and $\hat{K}_a$ are covariantly constant, and therefore commute with derivatives. The projection P_{ab} therefore does as well, and h^{TT}_{ab} satisfies the field equation $\Box h^{TT}_{ab} = \Box h_{ab}$. To check that it is in transverse-traceless gauge, we calculate

$$\hat{U}^a (P_a{}^m P_b{}^n - \tfrac{1}{2} P_{ab} P^{mn}) = 0 = \hat{K}^a (P_a{}^m P_b{}^n - \tfrac{1}{2} P_{ab} P^{mn}),$$

whence $\hat{U}^a A^{TT}_{ab} = 0 = \hat{K}^a A^{TT}_{ab} = 0$, and

$$\eta^{ab} (P_a{}^m P_b{}^n - \tfrac{1}{2} P_{ab} P^{mn}) = P^{mn} - \tfrac{1}{2} \cdot 2 \cdot P^{mn} = 0$$

The first result arises because P_{ab} is a projector the second, because it projects into a two-dimensional space.

b) Here, we note first that

$$P_{ab} = \bar{\eta}_{ab} - \hat{k}_a \hat{k}_b$$

Meanwhile, the trace part of A_{mn} is clearly killed in the tensor projection, so

$$[P_a{}^m P_b{}^n - \tfrac{1}{2} P_{ab} P^{mn}] =$$

$$= [P_a{}^i P_b{}^j - \tfrac{1}{2} P_{ab} P^{ij}][\bar{\eta}_i{}^m \bar{\eta}_j{}^n - \tfrac{1}{3} \bar{\eta}_{ij} \bar{\eta}^{mn}]$$

$$= [(\bar{\eta}_a{}^i - \hat{k}_a \hat{k}^i)(\bar{\eta}_b{}^j - \hat{k}_b \hat{k}^j)$$

$$- \tfrac{1}{2}(\bar{\eta}_{ab} - \hat{k}_a \hat{k}_b)(\bar{\eta}^{ij} - \hat{k}^i \hat{k}^j)][\bar{\eta}_i{}^m \bar{\eta}_j{}^n - \tfrac{1}{3} \bar{\eta}_{ij} \bar{\eta}^{mn}]$$

$$= [\bar{\eta}_a{}^i \bar{\eta}_j{}^n - \hat{k}_a \hat{k}^i \bar{\eta}_b{}^j - \hat{k}_b \hat{k}^j \bar{\eta}_a{}^i + \hat{k}_a \hat{k}_b \hat{k}^i \hat{k}^j$$

$$+ \tfrac{1}{2}(\bar{\eta}_{ab} - \hat{k}_a \hat{k}_b) \hat{k}^i \hat{k}^j][\bar{\eta}_i{}^m \bar{\eta}_j{}^n - \tfrac{1}{3} \bar{\eta}_{ij} \bar{\eta}^{mn}]$$

The result follows by combining the last two terms in the first bracket and noting that A_{mn} is symmetric.

3 The geodesic deviation equation derived in class for linearized fields gives

$$a_a = \tfrac{1}{2} \ddot{h}^{TT}_{ab} \xi^b$$

Here, a_a is the first-order relative acceleration of a pair of test masses that are at rest in the Newtonian frame of the Minkowski background, separated by the spatial displacement ξ^b. The right side is already first-order because of the metric perturbation, so we can use the zeroth-order approximant for ξ^b, which is constant. Since the metric perturbation is harmonic,

$$\ddot{a}_a = \frac{1}{2} \ddddot{h}^{TT}_{ab} \xi^b = -\frac{1}{2} \omega^2 \ddot{h}^{TT}_{ab} \xi^b = -\omega^2 a_a$$

This shows that the relative motion is also harmonic, with frequency ω, apart from two constants of integration corresponding to the initial relative displacement and velocity of the particles. The former is ξ^a, while the latter vanishes.

We can determine the radius of motion by computing the norm of a_a:

$$a_a = \frac{1}{2} \ddot{h}^{TT}_{ab} \xi^b = -\frac{1}{2} \omega^2 A \,\mathrm{Re}\left[e^{\circlearrowleft}_{ab} e^{-i\omega t}\right] \xi^b$$

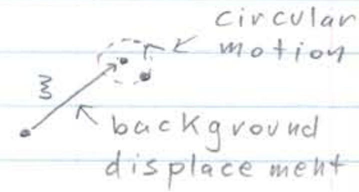
$$= -\frac{1}{2} \omega^2 A d \,\mathrm{Re}\left[e^{-i\omega t} \cdot \frac{1}{\sqrt{2}} \left(\hat{x}_a \hat{x}_b - \hat{y}_a \hat{y}_b + 2i \hat{x}_{(a} \hat{y}_{b)}\right) \hat{x}^b\right]$$

$$= -\frac{1}{2\sqrt{2}} \omega^2 A d \,\mathrm{Re}\left[e^{-i\omega t} \left(\hat{x}_a + i \hat{y}_a\right)\right]$$

$$= -\frac{1}{2\sqrt{2}} \omega^2 A d \left(\hat{x}_a \cos \omega t + \hat{y}_a \sin \omega t\right)$$

$$\Rightarrow \|a\| = \frac{1}{2\sqrt{2}} \omega^2 A d = \omega^2 r \quad \Rightarrow \quad r = \frac{Ad}{2\sqrt{2}}$$

Thus, the radius of the motion scales with the amplitude A of the wave, which of course makes sense.



4 a) The source energy density is

$$\rho(t,x,y,z) = m\,\delta(z)\left[\delta(x - a\cos\omega t)\,\delta(y - a\sin\omega t) + \delta(x + a\cos\omega t)\,\delta(y + a\sin\omega t)\right]$$

Therefore, the temporal metric perturbation is

$$h^{00} \cong \frac{4}{r}\int t^{00}(y)\,d^3y = \frac{8m}{r}$$

The temporal spatial components vanish;

$$\bar{h}^{0a} \cong \frac{4}{r}\int \bar{t}^{0a}(y)\,d^3y = \frac{4}{r}\frac{d}{dt}\int t^{00}\,\bar{y}^a\,d^3y = 0$$

because of the symmetry of the source. Finally the spatial components are

$$\bar{h}^{ab} \cong \frac{4}{r}\int \bar{t}^{ab}(y)\,d^3y = \frac{2}{r}\frac{d^2}{dt^2}\int t^{00}(y)\,\bar{y}^a\bar{y}^b\,d^3y$$

$$= \frac{2}{r}\frac{d^2}{dt^2}\,m\begin{pmatrix} 2a^2\cos\omega t & 2a^2\sin\omega t\cos\omega t & 0 \\ 2a^2\sin\omega t\cos\omega t & 2a^2\sin^2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{2ma^2}{r}\frac{d^2}{dt^2}\begin{pmatrix} 1+\cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & 1-\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Taking the derivatives now gives the result. Recall that we have been using retarded time implicitly throughout.

b) The Coulombic, temporal component vanishes under the spatial projection here. The remaining terms are already orthogonal to $\hat{k}_a = \hat{z}_a$, so the result follows directly.

c) Here, we must project, using problem 2:

$$h^{TT}_{ab} = \left[\bar{\eta}^m_a \bar{\eta}^n_b - 2\hat{x}^m \bar{\eta}^n_{(a} \hat{x}_{b)} + \tfrac{1}{2}(\bar{\eta}_{ab} + \hat{x}_a \hat{x}_b)\hat{x}^m \hat{x}^n\right]$$

$$\cdot - \frac{8\sqrt{2}\, m a^2 \omega^2}{r} \mathrm{Re}\left[e^{-2i\omega(t-r)} e^{\sigma z}_{mn}\right]$$

$$= -\frac{8\pi\sqrt{2}\, m a \omega^2}{r} \mathrm{Re}\Big[e^{-2i\omega(t-r)} \Big(e^{\sigma z}_{ab}$$

$$- 2\hat{x}^n e^{\sigma z}_{m(a} \hat{x}_{b)} + \tfrac{1}{2}(\bar{\eta}_{ab} + \hat{x}_a \hat{x}_b)\hat{x}^m \hat{x}^n e^{\sigma z}_{mn}\Big)\Big]$$

$$= -\frac{8\pi m a \omega^2}{r} \mathrm{Re}\Big[e^{-2i\omega(t-r)} \Big(\hat{x}_a \hat{x}_b - \hat{y}_a \hat{y}_b + 2i\hat{x}_{(a}\hat{y}_{b)}$$

$$- 2(\hat{x}_{(a} + i\hat{y}_{(a})\hat{x}_{b)} + \tfrac{1}{2}(2\hat{x}_a \hat{x}_b + \hat{y}_a \hat{y}_b + \hat{z}_a \hat{z}_b)\Big)\Big]$$

$$= -\frac{8\pi m a \omega^2}{r} \mathrm{Re}\left[e^{-2i\omega(t-r)} \left(-\tfrac{1}{2}\hat{y}_a \hat{y}_b + \tfrac{1}{2}\hat{z}_a \hat{z}_b\right)\right]$$

The final factor in braces is $-\frac{1}{2} e^{+\times}_{ab}$, and the result then follows immediately.