Problem Set III

1 The de Donder gauge equation is

$$\Box h_{ab} = -16\pi T_{ab} = -16\pi \left[2\hat{t}_{(a} p_{b)} + (\hat{t}^c p_c)\hat{t}_a \hat{t}_b \right]$$

The vector $\hat{t}^b$ commutes with the D'Alambertian, so we find

$$\Box (h_{ab}\hat{t}^b) = -16\pi \left[\hat{t}_a p_b \hat{t}^b + p_a \hat{t}_b \hat{t}^b + \hat{t}^c p_c \hat{t}_a \hat{t}_b \hat{t}^b \right]$$

$$= -16\pi \left[p_a \cdot -1 \right] = 16\pi p_a = -4\pi \cdot -4 p_a$$

The result follows immediately. We also have

$$\Box \left[8\hat{t}_{(a} A_{b)} + 4(\hat{t}^c A_c)\hat{t}_a \hat{t}_b \right]$$

$$= 8\hat{t}_{(a} \Box A_{b)} + 4(\hat{t}^c \Box A_c)\hat{t}_a \hat{t}_b$$

$$= -32\pi \hat{t}_{(a} p_{b)} - 16\pi (\hat{t}^c p_c)\hat{t}_a \hat{t}_b = -16\pi T_{ab}$$

2 To first order, we can take the 4-velocity of a test body to obey

$$\frac{dx^0}{d\tau} = 1 \quad \text{and} \quad \frac{dx^i}{d\tau} = v^i \quad i = 1, 2, 3$$

and $d\tau = dt$. Therefore, again to first order,

$$\frac{d^2 x^\gamma}{d\tau^2} - \Gamma_{\alpha\beta}{}^{\gamma} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\Rightarrow \frac{d^2 x^i}{dt^2} = \Gamma_{00}{}^{i} + 2\Gamma_{j0}{}^{i} \frac{dx^j}{dt}$$

Thus, we calculate

$$\Gamma_{00}{}^{i} = -\tfrac{1}{2} g^{i\alpha}\left(2\partial_{(0}\, g_{0)\alpha} - \partial_\alpha g_{00}\right)$$

$$\Gamma_{j0}{}^{i} = -\tfrac{1}{2} g^{i\alpha}\left(2\partial_{(j}\, g_{0)\alpha} - \partial_\alpha g_{j0}\right)$$

To get the metric components, we write

$$\dot{g}_{ab} = h_{ab} - \tfrac{1}{2}\eta_{ab} h$$

$$= 8\,\hat{t}_{(a} A_{b)} + 4(\hat{t}^c A_c)\,\hat{t}_a \hat{t}_b - \tfrac{1}{2}\eta_{ab}\cdot 4\hat{t}^c A_c$$

$$= 8\,\hat{t}_{(a} A_{b)} + 6\,(\hat{t}^c A_c)\,\hat{t}_a \hat{t}_b - 2(\hat{t}^c A_c)\,\sigma_{ab}$$

$$\Rightarrow ds^2 = -(1+2A_0)\,dt^2 - 8A_j\, dt\, x^j$$

$$+ (1-2A_0)(dx^2+dy^2+dz^2)$$

Neglecting time derivatives of the fields,

$$\Gamma_{00}{}^{i} = -\tfrac{1}{2}\partial^i(-1-2A_0) = \partial^i A_0$$

$$\Gamma_{j0}{}^{i} = -\tfrac{1}{2} g^{ii}\left(\partial_j\cdot -4A_i - \partial_i\cdot -4A_j\right) = 2(\partial_j A^i - \partial^i A_j)$$

Thus, the geodesic equation is

$$\ddot{x}^i = \partial^i A_0 + 4\dot{x}^j(\partial_j A^i - \partial^i A_j)$$

$$\varepsilon_{ijk} v^j B^k = \varepsilon_{ijk}\,\varepsilon^{klm} v^j \partial_l A_m = 2\delta_i^{[l}\,\delta_j^{m]}\, v^j \partial_l A_m$$

$$= v^j(\partial_i A_j - \partial_j A_i)$$

Thus, the last term here is indeed $-4\vec{\nabla}\times(\vec{\nabla}\times\vec{A})$. The result follows.

3 We do this by calculating the field from the corresponding electromagnetic source.

$$p_a = \underset{M/4\pi R^2}{\sigma}\ \delta_R(r)\left[\underset{\text{mass density}}{\hat{t}_a} + \underset{\vec{\omega}\times\vec{r}\text{ (spatial) rotation velocity}}{\varepsilon_{abc}\,\omega^b r^c}\right]$$

We can ignore relativistic effects (e.g. contraction) here because they are at least second-order.

Now, the magnetic field here is given by

$$\vec{B}(\vec{r}) = \int \vec{J}(\vec{r}\,')\times\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}\,d^3r^3$$

$$= \int \sigma\,\delta_R(r')\,(\vec{\omega}\times\vec{r}\,')\times\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|}\,d^3r'$$

$$= \sigma\int\delta_R(r')\left[\vec{r}\,'\,\frac{\vec{\omega}\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} - \vec{\omega}\,\frac{\vec{r}\,'\cdot(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right]d^3r'$$

This suggests we study the tensor integral

$$\int\delta_R(r')\,\frac{\vec{r}\,'(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,d^3r' = \int\delta_R(r')\,\vec{r}\,'\,\vec{\nabla}\,\frac{-1}{|\vec{r}-\vec{r}\,'|}\,d^3r'$$

$$= -\vec{\nabla}\oint\frac{R\hat{r}'}{|\vec{r}-R\hat{r}'|}\,R^2\,d\Omega_{\hat{r}'}$$

$$= -R^3\,\vec{\nabla}\int_0^{\pi}\frac{2\pi\,\hat{r}\cos\theta}{|\vec{r}-R\hat{r}'|}\sin\theta\,d\theta$$

$$= -2\pi R^3\,\vec{\nabla}\left(\hat{r}\int_{-1}^{1}\frac{z\,dz}{\sqrt{r^2+R^2-2rRz}}\right) = -2\pi R^3\,\vec{\nabla}\left(\hat{r}\,\frac{2r}{3R^2}\right)$$

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The gradient of $\vec{r}$ gives the identity tensor, so we find

$$\int \delta_R(r') \frac{\vec{r}\,'(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} d^3r' = -\frac{4\pi}{3} R \cdot \mathrm{id}$$

Therefore, we get

$$\vec{B}(\vec{r}) = -\frac{4\pi}{3} R\sigma \left[\mathrm{id}\cdot\vec{\omega} - \vec{\omega}\ \mathrm{tr}(\mathrm{id}) \right]$$

$$= \frac{8\pi}{3} R\sigma\vec{\omega} = \frac{8\pi}{3} R \frac{M}{4\pi R^2} \vec{\omega} = \frac{2M}{3R} \vec{\omega}$$

The electric field vanishes because we are inside a uniform shell and the potential is therefore constant.

The precession effect is given by

$$0 = \nabla_{\hat{t}} s^i = \partial_{\hat{t}} s^i - \hat{t}^a s^b \Gamma_{ab}{}^i$$

$$\Rightarrow \dot{s}^i = \Gamma_{0j}{}^i s^j = 2(\partial_j A^i - \partial^i A_j) s^j$$

$$\Rightarrow \dot{\vec{s}} = -2\vec{s}\times\vec{B} = 2\vec{B}\times\vec{s}$$

This is the result.

4 a) This follows from the standard result that the inner product of a Killing field ξ with an affine geodetic field η is constant along each geodesic curve:

$$\nabla_\eta(\eta\cdot\xi) = \nabla_\eta\eta\cdot\xi + \eta\cdot\nabla_\eta\xi = \eta^a\eta^b\nabla_a\xi_b = 0$$

b) The four-velocities of the atoms in this problem, which are static relative to the sun, must be <u>normalized</u>:

$$\hat{u}_e = \left(1 - \frac{2M}{r_e}\right)^{-1/2} \partial_t$$

$$\hat{u}_r = \left(1 - \frac{2M}{r_r}\right)^{-1/2} \partial_t$$

Therefore, we find

$$\left(1 - \frac{2M}{r_r}\right)^{1/2} w_r = \left(1 - \frac{2M}{r_r}\right)^{1/2} \hat{u}_r \cdot \eta = \partial_t \cdot \eta$$

$$= \left(1 - \frac{2M}{r_e}\right)^{1/2} \hat{u}_e \cdot \eta = \left(1 - \frac{2M}{r_e}\right)^{1/2} w_e$$

$$\Rightarrow \quad w_e = \left(1 - \frac{2M}{r_e}\right)^{-1/2} \left(1 - \frac{2M}{r_r}\right)^{1/2} w_r$$

$$\cong \left(1 - \frac{2M}{R}\right)^{-1/2} w_r \cong \left(1 + \frac{M}{R}\right) w_r$$

$$\Rightarrow \quad \frac{w_e}{w_r} \cong 1 + \frac{M}{R} \rightsquigarrow z \cong \frac{M}{R} \approx 2.12 \times 10^{-6}$$

We calculated the numerical result in class to get the deflection of light.

5 The gauge transformation of $\dot{g}_{ab}$ is

$$\dot{g}_{ab} \mapsto \dot{\tilde{g}}_{ab} = \dot{g}_{ab} + 2 \nabla_{(a} \phi_{b)}$$

Meanwhile, we have

$$\dot{\nabla}_{ab}{}^{c} = -\frac{1}{2} g^{cm} \left(2 \nabla_{(a} \dot{g}_{b)m} - \nabla_m \dot{g}_{ab}\right) = -\frac{1}{2} g^{cm} \left(\nabla_a \dot{g}_{bm} + 2 \nabla_{[b} \dot{g}_{m]a}\right)$$

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Under a gauge transformation, therefore,

$$\dot{\nabla}_{ab}{}^{c} \mapsto \dot{\tilde{\nabla}}_{ab}{}^{c} = \dot{\nabla}_{ab}{}^{c} - \tfrac{1}{2} g^{cm}\big(2\nabla_a \nabla_{(b}\phi_{m)} + 2\nabla_{[b}\nabla_{m]}\phi_a + 2\nabla_{[b}\nabla_{|a|}\phi_{m]}\big)$$

$$= \dot{\nabla}_{ab}{}^{c} - \tfrac{1}{2} g^{cm}\big(2\nabla_a\nabla_{(b}\phi_{m)} + R_{bma}{}^{n}\phi_n + 2\nabla_a\nabla_{[b}\phi_{m]} + 2R_{[b|a|m]}{}^{n}\phi_n\big)$$

$$= \dot{\nabla}_{ab}{}^{c} - \tfrac{1}{2} g^{cm}\big(2\nabla_a\nabla_b\phi_m + R_{bma}{}^{n}\phi_n + R_{bam}{}^{n}\phi_n - R_{mab}{}^{n}\phi_n\big)$$

$$= \dot{\nabla}_{ab}{}^{c} - \nabla_a\nabla_b\phi^c - \tfrac{1}{2} g^{cm}\big(R_{bman} - R_{abmn} - R_{mabn}\big)\phi^n$$

$$= \dot{\nabla}_{ab}{}^{c} - \nabla_a\nabla_b\phi^c - \tfrac{1}{2} g^{cm}\big(2R_{bman}\big)\phi^n$$

$$= \dot{\nabla}_{ab}{}^{c} - \nabla_a\nabla_b\phi^c - \phi^n R_{anb}{}^{c}$$

$$= \dot{\nabla}_{ab}{}^{c} + \phi^m R_{mab}{}^{c} - \nabla_a\nabla_b\phi^c$$

6 a) The Riemann perturbation is therefore

$$\dot{\tilde{R}}_{abc}{}^{d} = 2\nabla_{[a}\dot{\tilde{\nabla}}_{b]c}{}^{d}$$

$$= 2\nabla_{[a}\dot{\nabla}_{b]c}{}^{d} + 2\nabla_{[a}\big(\phi^m R_{|m|b]c}{}^{d}\big) - 2\nabla_{[a}\nabla_{b]}\nabla_c\phi^d$$

$$= \dot{R}_{abc}{}^{d} + \phi^m \cdot 2\nabla_{[a}R_{|m|b]c}{}^{d} + R_{mbc}{}^{d}\nabla_a\phi^m$$

$$- R_{mac}{}^{d}\nabla_b\phi^m - R_{abc}{}^{m}\nabla_m\phi^c + R_{abm}{}^{d}\nabla_c\phi^m$$

Using the second Bianchi identity on the first term leaves

$$\dot{\tilde{R}}_{abc}{}^{d} = \dot{R}_{abc}{}^{d} - \phi^m \left(\nabla_b R_{mac}{}^{d} + \nabla_a R_{bmc}{}^{d} \right)$$

$$+ R_{mbc}{}^{d} \nabla_a \phi^m + R_{amc}{}^{d} \nabla_a \phi^m$$

$$+ R_{abm}{}^{d} \nabla_c \phi^m - R_{abc}{}^{m} \nabla_m \phi^d$$

$$= \dot{R}_{abc}{}^{d} + \phi^m \nabla_m R_{abc}{}^{d}$$

$$+ R_{mbc}{}^{d} \nabla_a \phi^m + R_{amc}{}^{d} \nabla_a \phi^m$$

$$+ R_{abm}{}^{d} \nabla_c \phi^m - R_{abc}{}^{m} \nabla_m \phi^d .$$

$$= \dot{R}_{abc}{}^{d} + \mathcal{L}_\phi R_{abc}{}^{d}$$

b) The result for R_{ac} follows immediately from part (a) because $\delta_d^{\,b}$ commutes with both $\frac{d}{d\lambda}$ and $\mathcal{L}_\phi$. Then, we have

$$\dot{\tilde{G}}_{ab} = \frac{d}{d\lambda} \left[\left(\delta_a^m \delta_b^n - \tfrac{1}{2} \tilde{g}_{ab} \tilde{g}^{mn} \right) \tilde{R}_{mn} \right]$$

$$= \left(\delta_a^m \delta_b^n - \tfrac{1}{2} \tilde{g}_{ab} \tilde{g}^{mn} \right) \dot{\tilde{R}}_{mn}$$

$$- \tfrac{1}{2} \left(\dot{\tilde{g}}_{ab} \tilde{g}^{mn} - \tilde{g}_{ab} \dot{\tilde{g}}^{mn} \right) \tilde{R}_{mn}$$

$$= \left(\delta_a^m \delta_b^n - \tfrac{1}{2} \mathring{g}_{ab} \mathring{g}^{mn} \right) \left(\dot{R}_{mn} + \mathcal{L}_\phi \mathring{R}_{mn} \right)$$

$$- \tfrac{1}{2} \mathring{g}^{mn} \left(\dot{g}_{ab} + \mathcal{L}_\phi \mathring{g}_{ab} \right) \mathring{R}_{mn}$$

$$+ \tfrac{1}{2} \mathring{g}_{ab} \left(\dot{g}_{mn} + \mathcal{L}_\phi \mathring{g}_{mn} \right) \mathring{R}^{mn}$$

The first terms on each line combine to give $\dot{G}_{ab}$. Meanwhile, we have

$$(\delta^m_a \delta^n_b - \tfrac{1}{2} \mathring{g}_{ab} \mathring{g}^{mn}) \mathcal{L}_\phi \mathring{R}_{mn}$$

$$= \mathcal{L}_\phi \mathring{G}_{ab} + \tfrac{1}{2} \mathring{R}_{mn} \mathcal{L}_\phi (\mathring{g}_{ab} \mathring{g}^{mn})$$

$$= \mathcal{L}_\phi \mathring{G}_{ab} + \tfrac{1}{2} \mathring{R} \mathcal{L}_\phi \mathring{g}_{ab} + \tfrac{1}{2} \mathring{R}_{mn} \mathring{g}_{ab} \mathcal{L}_\phi \mathring{g}^{mn}$$

$$= \mathcal{L}_\phi \mathring{G}_{ab} + \tfrac{1}{2} \mathring{R} \mathcal{L}_\phi \mathring{g}_{ab} - \tfrac{1}{2} \mathring{R}^{mn} \mathring{g}_{ab} \mathcal{L}_\phi \mathring{g}_{mn}$$

The terms from differentiating by parts cancel the extra terms on the second two lines on the previous page. The result $\dot{\tilde{G}}_{ab} = \dot{G}_{ab} + \mathcal{L}_\phi \mathring{G}_{ab}$ then follows.

c) Since $G_{ab}(\lambda)$ is a tensor, it obeys the same tensorial transformation law

$$G_{ab}(\lambda) \mapsto \tilde{G}_{ab}(\lambda) = \Phi(\lambda) \cdot G_{ab}(\lambda)$$

that the metric does under a diffeomorphism. Taking $\frac{d}{d\lambda}$ at $\lambda = 0$ gives

$$\dot{\tilde{G}}_{ab}(0) = \Phi(0) \cdot \dot{G}_{ab}(0) + \mathcal{L}_\phi G_{ab}(0)$$

This is the result.

d) The source $T_{ab}(\lambda)$ is also a tensor, so

$$\dot{\tilde{G}}_{ab} = \dot{G}_{ab} + \mathcal{L}_\phi \mathring{G}_{ab} = 8\pi \dot{T}_{ab} + \mathcal{L}_\phi (8\pi \mathring{T}_{ab})$$