Lecture 1

The Schwarzschild Metric

1.1 THE CURVATURE OF STATIC SPHERICAL SPACETIMES

The most general static, spherically symmetric metric can be written in the form

$$\mathrm{d}s^2 = -N^2(r)\,\mathrm{d}t^2 + F^{-2}(r)\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2, \tag{1.1}$$

provided we assume that the one-form $\mathrm{d}r$ vanishes nowhere. The coordinate t here is unique static time coordinate, normalized so that its derivative along the static Killing field is unity, while the coordinate r is the areal radius. We can easily read off from this the orthonormal basis

$$e^\alpha = \begin{pmatrix} N\,\mathrm{d}t \\ F^{-1}\,\mathrm{d}r \\ r\,\mathrm{d}\theta \\ r\sin\theta\,\mathrm{d}\phi \end{pmatrix} \tag{1.2}$$

of co-vector fields. The concrete index α here takes the values t, r, θ, ϕ, and we have written the corresponding basis fields in that order in the column. Since the dual basis is diagonal in the sense that each basis field e^α is proportional to the gradient $\mathrm{d}x^\alpha$ of the corresponding coordinate, it is easy to read off the vector basis

$$e_\alpha = \begin{pmatrix} N^{-1}\,\partial_t \\ F\,\partial_r \\ r^{-1}\,\partial_\theta \\ r^{-1}\csc\theta\,\partial_\phi \end{pmatrix}. \tag{1.3}$$

One can verify immediately that $e_\alpha(e^\beta) = \delta^\beta_\alpha$ here. Furthermore, the matrix of inner products is also clearly

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \tag{1.4}$$

This confirms that (1.2) is indeed an orthonornal basis.

Our goal is to calculate the Einstein tensor for metrics of the form (1.1), so that we may solve the Einstein equation. We do this in the Cartan approach. Conceptually, the first step to calculating the curvature of spacetime is to find the metric connection ∇_a whose curvature it is. In Cartan's approach, we do this by solving the relation

$$\eta_{\alpha\beta}\,\mathrm{d}e^{\beta} = \begin{pmatrix} -N'\,\mathrm{d}r\wedge\mathrm{d}t \\ 0 \\ \mathrm{d}r\wedge\mathrm{d}\theta \\ \sin\theta\,\mathrm{d}r\wedge\mathrm{d}\phi + r\cos\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi \end{pmatrix} = -\omega_{\alpha\beta}\wedge\begin{pmatrix} N\,\mathrm{d}t \\ F^{-1}\,\mathrm{d}r \\ r\,\mathrm{d}\theta \\ r\sin\theta\,\mathrm{d}\phi \end{pmatrix} \tag{1.5}$$

for the unique *anti-symmetric* matrix $\omega_{\alpha\beta}$ of Cartan connection forms. While the question of the Cartan matrix does have a unique answer, the process to find that answer is a bit more an art than a science.

Exercise 1.6: Determine the matrix $\omega_{\alpha\beta}$ of Cartan connection forms from (1.5).

Solution: To illustrate how this process works in general, let us first write the Cartan matrix in the completely generic form

$$-\omega_{\alpha\beta} = \begin{pmatrix} 0 & \{tr\theta\phi\} & \{tr\theta\phi\} & \{tr\theta\phi\} \\ -\{tr\theta\phi\} & 0 & \{tr\theta\phi\} & \{tr\theta\phi\} \\ -\{tr\theta\phi\} & -\{tr\theta\phi\} & 0 & \{tr\theta\phi\} \\ -\{tr\theta\phi\} & -\{tr\theta\phi\} & -\{tr\theta\phi\} & 0 \end{pmatrix}, \tag{1.6a}$$

where the notation $\{tr\theta\phi\}$ means that there is an unknown one-form with all four components undetermined in each of these slots. We have used the antisymmetry of $\omega_{\alpha\beta}$ to set the diagonal terms to zero.

Let's focus now on the first row of (1.5). We need to get a term proportional to $\mathrm{d}t\wedge\mathrm{d}r$, and this could come either from a term proportional to $\mathrm{d}r$ in the top left entry of $\omega_{\alpha\beta}$, or from a term proportional to $\mathrm{d}t$ in the term to its right. Since the top left entry vanishes by anti-symmetry, only the latter option remains, so

$$-\omega_{\alpha\beta} = \begin{pmatrix} 0 & FN'\,\mathrm{d}t + \{r\theta\phi\} & \{r\theta\phi\} & \{r\theta\phi\} \\ -FN'\,\mathrm{d}t - \{r\theta\phi\} & 0 & \{tr\theta\phi\} & \{tr\theta\phi\} \\ -\{r\theta\phi\} & -\{tr\theta\phi\} & 0 & \{tr\theta\phi\} \\ -\{r\theta\phi\} & -\{tr\theta\phi\} & -\{tr\theta\phi\} & 0 \end{pmatrix}. \tag{1.6b}$$

We have also used the absence of terms proportional to $\mathrm{d}t\wedge\mathrm{d}\theta$ or $\mathrm{d}t\wedge\mathrm{d}\phi$ in the first row on the left side of (1.5) to eliminate terms proportional to $\mathrm{d}t$ in the remaining entries in the first row of $\omega_{\alpha\beta}$. We then used anti-symmetry to simplify the first column accordingly.

Now apply the same trick to the other rows of $\omega_{\alpha\beta}$: the only way to get two-forms proportional to $\mathrm{d}x^{\alpha}$ in row α is if they come from $\omega_{\alpha\beta}$ with *other* values of β. This lets us read off

$$-\omega_{\alpha\beta} = \begin{pmatrix} 0 & FN'\,\mathrm{d}t + \{\theta\phi\} & \{r\theta\phi\} & \{r\theta\phi\} \\ -FN'\,\mathrm{d}t - \{\theta\phi\} & 0 & \{t\theta\phi\} & \{t\theta\phi\} \\ -\{r\phi\} & -F\,\mathrm{d}\theta - \{tr\phi\} & 0 & \{tr\phi\} \\ -\{r\theta\} & -F\sin\theta\,\mathrm{d}\phi - \{tr\theta\} & -\cos\theta\,\mathrm{d}\phi - \{tr\theta\} & 0 \end{pmatrix}. \tag{1.6c}$$

Then reassert anti-symmetry to simplify further:

$$-\omega_{\alpha\beta} = \begin{pmatrix} 0 & FN'\,\mathrm{d}t + \{\theta\phi\} & \{r\phi\} & \{r\theta\} \\ -FN'\,\mathrm{d}t - \{\theta\phi\} & 0 & F\mathrm{d}\theta + \{t\phi\} & F\sin\theta\,\mathrm{d}\phi + \{t\theta\} \\ -\{r\phi\} & -F\,\mathrm{d}\theta - \{t\phi\} & 0 & \cos\theta\,\mathrm{d}\phi + \{tr\} \\ -\{r\theta\} & -F\sin\theta\,\mathrm{d}\phi - \{t\theta\} & -\cos\theta\,\mathrm{d}\phi - \{tr\} & 0 \end{pmatrix}. \tag{1.6d}$$

This ends the easy part of the calculation.

There are obviously many undetermined components remaining in (1.6d), and these are related to one another in various ways. For example, the coefficient of $\mathrm{d}r$ in the top right entry $\omega_{t\phi}$ here must cancel against the coefficient of $\mathrm{d}\phi$ in ω_{tr} two columns to its left since the $\mathrm{d}t$ term in ω_{tr} already gives us the correct result on the left side of (1.5). This, in turn, dictates the coefficient of $\mathrm{d}\phi$ in ω_{rt} by anti-symmetry, which then must cancel against the coefficient of $\mathrm{d}t$ in $\omega_{r\phi}$ three columns to its right. This determines the coefficient of $\mathrm{d}t$ in $\omega_{\phi r}$ by anti-symmetry, and thence the coefficient of $\mathrm{d}r$ in $\omega_{\phi t}$, which determines the coefficient of $\mathrm{d}r$ in $\omega_{t\phi}$ by anti-symmetry. We are now back where we started, with the $\mathrm{d}r$ part of $\omega_{r\phi}$, and rearranging terms should yield a soluble equation for the $\mathrm{d}r$ part of ω_{tr} since the Cartan matrix is unique. In fact, since each equation in this chain was a simple proportionality, the unique solution must be zero, and all of the intermediate expressions must vanish as well.

But there is an easier way to finish this calculation, which is simply to *guess* the right answer. Indeed, it is striking that we already have known terms in (1.6d) that suffice to give the *entire* left side of (1.5). Motivated by this, we check that

$$\begin{pmatrix} 0 & FN'\,\mathrm{d}t & 0 & 0 \\ -FN'\,\mathrm{d}t & 0 & F\,\mathrm{d}\theta & F\sin\theta\,\mathrm{d}\phi \\ 0 & -F\,\mathrm{d}\theta & 0 & \cos\theta\,\mathrm{d}\phi \\ 0 & -F\sin\theta\,\mathrm{d}\phi & -\cos\theta\,\mathrm{d}\phi & 0 \end{pmatrix} \wedge \begin{pmatrix} N\,\mathrm{d}t \\ F^{-1}\,\mathrm{d}r \\ r\,\mathrm{d}\theta \\ r\sin\theta\,\mathrm{d}\phi \end{pmatrix}$$

$$= \begin{pmatrix} N'\,\mathrm{d}t\wedge\mathrm{d}r \\ 0 \\ -\mathrm{d}\theta\wedge\mathrm{d}r \\ -\sin\theta\,\mathrm{d}\phi\wedge\mathrm{d}r - r\cos\theta\,\mathrm{d}\phi\wedge\mathrm{d}\theta \end{pmatrix}. \tag{1.6e}$$

The right side here is exactly the left side of (1.5) due to the anti-symmetry of the wedge product. Thus, the since the Cartan matrix is unique, $-\omega_{\alpha\beta}$ must equal the matrix on the left side of (1.6e).

Comment: In general, one does not use the notation $\{tr\theta\phi\}$ to indicate the presence of undetermined components of forms in the Cartan process. It is generally better to "tinker," filling in entries using the first principle above—that the only way to get two-forms proportional to $\mathrm{d}x^\alpha$ in row α of $\mathrm{d}e^\alpha$ is if they come from other values of β in the matrix $\omega_{\alpha\beta}$—and then imposing anti-symmetry. In general, this will not produce the answer immediately, as it has here, but some additional corrections to other entries will be needed to compensate for terms brought in by anti-symmetrization. Generally, however, an answer emerges after only one or two more iterations. One would be wise, however, always to check the final result, as we have done in (1.6e).

Exercise 1.7: Calculate the matrix of Cartan connection forms directly from the formula

$$C_{abc} = T_{a[bc]} - \tfrac{1}{2}\,T_{bca} \tag{1.7a}$$

for the difference tensor relating the metric connection ∇_a to the basis connection D_a associated with the orthonormal basis e^a_α. Here, T_{abc} denotes the torsion

$$T_{ab}{}^c = -e^\alpha_a\,e^\beta_b\,[e_\alpha, e_\beta]^c \tag{1.7b}$$

of D_a, with its last index lowered using the metric.

Solution: The components $T_{\alpha\beta}{}^c$ of the torsion tensor are given simply by the matrix

$$-T(e_\alpha, e_\beta) = [e_\alpha, e_\beta] = \begin{pmatrix} [e_t, e_t] & [e_t, e_r] & [e_t, e_\theta] & [e_t, e_\phi] \\ [e_r, e_t] & [e_r, e_r] & [e_r, e_\theta] & [e_r, e_\phi] \\ [e_\theta, e_t] & [e_\theta, e_r] & [e_\theta, e_\theta] & [e_\theta, e_\phi] \\ [e_\phi, e_t] & [e_\phi, e_r] & [e_\phi, e_\theta] & [e_\phi, e_\phi] \end{pmatrix} \tag{1.7c}$$

of Lie brackets of the basis vector fields. Note that we have written this so that the first index, α, runs down the columns the matrix while the second index, β, runs across the rows. These Lie brackets are all easy enough to calculate:

$$T(e_\alpha, e_\beta) = \begin{pmatrix} 0 & -FN^{-2}N'\,\partial_t & 0 & 0 \\ FN^{-2}N'\,\partial_t & 0 & Fr^{-2}\,\partial_\theta & Fr^{-2}\csc\theta\,\partial_\phi \\ 0 & -Fr^{-2}\,\partial_\theta & 0 & r^{-2}\csc\theta\cot\theta\,\partial_\phi \\ 0 & -Fr^{-2}\csc\theta\,\partial_\phi & -r^{-2}\csc\theta\cot\theta\,\partial_\phi & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -FN^{-1}N'\,e_t & 0 & 0 \\ FN^{-1}N'\,e_t & 0 & Fr^{-1}\,e_\theta & Fr^{-1}\,e_\phi \\ 0 & -Fr^{-1}\,e_\theta & 0 & r^{-1}\cot\theta\,e_\phi \\ 0 & -Fr^{-1}\,e_\phi & -r^{-1}\cot\theta\,e_\phi & 0 \end{pmatrix}. \tag{1.7d}$$

For our calculations, it will be more convenient to lower the remaining abstract index, c:

$$T_{\alpha\beta c} = \begin{pmatrix} 0 & FN^{-1}N'\,e^t_c & 0 & 0 \\ -FN^{-1}N'\,e^t_c & 0 & Fr^{-1}\,e^\theta_c & Fr^{-1}\,e^\phi_c \\ 0 & -Fr^{-1}\,e^\theta_c & 0 & r^{-1}\cot\theta\,e^\phi_c \\ 0 & -Fr^{-1}\,e^\phi_c & -r^{-1}\cot\theta\,e^\phi_c & 0 \end{pmatrix}. \tag{1.7e}$$

Note that the signature of the metric is important here because $g_{ab}\,e^b_t = -e^t_a$, while the spatial basis vectors suffer no such change of sign when we lower their abstract indices.

To calculate the Cartan matrix, we must find

$$\omega_{m\alpha\beta} = C_{mbc}\,e^b_\alpha\,e^c_\beta = T_{m[\alpha\beta]} - \tfrac{1}{2}\,T_{\alpha\beta m}. \tag{1.7f}$$

We have already found the second term on the right here, but must still calculate the first. We do this by making the first index, α, on the torsion abstract and the last index, c, concrete. This is straightforward to do in our matrix notation. We have, for instance,

$$T_{a\beta c} = e^\alpha_a\,T_{\alpha\beta c} = \begin{pmatrix} e^t_a & e^r_a & e^\theta_a & e^\phi_a \end{pmatrix}\,T_{\alpha\beta c}$$

$$= \begin{pmatrix} -FN^{-1}N'\,e^r_a\,e^t_c \\ FN^{-1}N'\,e^t_a\,e^t_c - Fr^{-1}\,e^\theta_a\,e^\theta_c - Fr^{-1}\,e^\phi_a\,e^\phi_c \\ Fr^{-1}\,e^r_a\,e^\theta_c - r^{-1}\cot\theta\,e^\phi_a\,e^\phi_c \\ Fr^{-1}\,e^r_a\,e^\phi_c + r^{-1}\cot\theta\,e^\theta_a\,e^\phi_c \end{pmatrix}^{\top}. \tag{1.7g}$$

We have chosen to write e^α_a as a row vector here because it then contracts naturally down the columns of the matrix above. The result of this matrix product is a row vector of contravariant tensors with two indices. We have written this as the transpose of a column vector to save space. When we make the last index concrete, we must find a matrix of one-forms. To do this, we take the matrix product of the *column* vector we just found with the *row* vector of basis vector fields e^c_γ, in that order:

$$T_{a\beta\gamma} = T_{a\beta c}\,\begin{pmatrix} e^c_t & e^c_r & e^c_\theta & e^c_\phi \end{pmatrix}$$

$$= \begin{pmatrix} -FN^{-1}N'\,e^r_a & 0 & 0 & 0 \\ FN^{-1}N'\,e^t_a & 0 & -Fr^{-1}\,e^\theta_a & -Fr^{-1}\,e^\phi_a \\ 0 & 0 & Fr^{-1}\,e^r_a & -r^{-1}\cot\theta\,e^\phi_a \\ 0 & 0 & 0 & Fr^{-1}\,e^r_a + r^{-1}\cot\theta\,e^\theta_a \end{pmatrix}. \tag{1.7h}$$

Note that this produces a matrix such that the *first* concrete index, β in this case, in the tensor expression on the left once again runs *down* the columns of the matrix. This will allow us to add this result to our previous one using the usual laws of matrix addition.

To complete our calculation, we now take the antisymmetric part of the previous result. Renaming indices, we write

$$T_{a[\alpha\beta]} = \frac{1}{2}\begin{pmatrix} 0 & -FN^{-1}N'\,e_a^t & 0 & 0 \\ FN^{-1}N'\,e_a^t & 0 & -Fr^{-1}\,e_a^\theta & -Fr^{-1}\,e_a^\phi \\ 0 & Fr^{-1}\,e_a^\theta & 0 & -r^{-1}\cot\theta\,e_a^\phi \\ 0 & Fr^{-1}\,e_a^\phi & r^{-1}\cot\theta\,e_a^\phi & 0 \end{pmatrix}. \tag{1.7i}$$

Remarkably, this tensor is exactly equal to the torsion tensor above. This certainly does not happen all the time. But in this case, it makes the Cartan matrix very easy to calculate:

$$\omega_{a\alpha\beta} = T_{a[\alpha\beta]} - \tfrac{1}{2}\,T_{\alpha\beta a} = \begin{pmatrix} 0 & -FN^{-1}N'\,e_a^t & 0 & 0 \\ FN^{-1}N'\,e_a^t & 0 & -Fr^{-1}\,e_a^\theta & -Fr^{-1}\,e_a^\phi \\ 0 & Fr^{-1}\,e_a^\theta & 0 & -r^{-1}\cot\theta\,e_a^\phi \\ 0 & Fr^{-1}\,e_a^\phi & r^{-1}\cot\theta\,e_a^\phi & 0 \end{pmatrix}. \tag{1.7j}$$

The remainder of the curvature caalculation will be easier if we translate back to differential-forms notation and suppress the abstract index on the result:

$$\omega_{\alpha\beta} = \begin{pmatrix} 0 & -FN'\,\mathrm{d}t & 0 & 0 \\ FN'\,\mathrm{d}t & 0 & -F\,\mathrm{d}\theta & -F\sin\theta\,\mathrm{d}\phi \\ 0 & F\,\mathrm{d}\theta & 0 & -\cos\theta\,\mathrm{d}\phi \\ 0 & F\sin\theta\,\mathrm{d}\phi & \cos\theta\,\mathrm{d}\phi & 0 \end{pmatrix}. \tag{1.7k}$$

This is the Cartan matrix we seek, and the result is identical to that of the previous exercise.

Comment: This approach to the problem of finding the Cartan matrix is entirely algorithmic. Presumably, something quite like it underlies the canned routines used by modern computer-algebra packages to calculate the Riemann tensor on spacetime. The directness of the calculation, however, is spoiled somewhat by the need to pay very close attention to the association of concrete indices with matrix entries. The previous approach required somewhat less worry of this sort.

To proceed, we must raise the second index on the Cartan matrix calculated in the previous exercise using the matrix of inverse metric components, which obviously has the same form as the metric-ccomponent matrix in (1.4). Since the β index in (1.6e) runs *across* the columns of the matrix, this is equivalent to multiplying (1.7k) by the metric matrix from the *right*:

$$\omega_\alpha{}^\beta = \omega_{\alpha\gamma}\,\eta^{\gamma\beta} = \begin{pmatrix} 0 & -FN'\,\mathrm{d}t & 0 & 0 \\ -FN'\,\mathrm{d}t & 0 & -F\,\mathrm{d}\theta & -F\sin\theta\,\mathrm{d}\phi \\ 0 & F\,\mathrm{d}\theta & 0 & -\cos\theta\,\mathrm{d}\phi \\ 0 & F\sin\theta\,\mathrm{d}\phi & \cos\theta\,\mathrm{d}\phi & 0 \end{pmatrix}. \tag{1.8}$$

This is the key result we need to calculate the curvature of a static, spherically symmetric spacetime.

The remainder of the calculation is quite mechanical. The two terms needed to calculate the Riemann curvature are the exterior derivative

$$\mathrm{d}\omega_\alpha{}^\beta = \begin{pmatrix} 0 & -(FN')'\,\mathrm{d}r\wedge\mathrm{d}t & 0 & 0 \\ -(FN')'\,\mathrm{d}r\wedge\mathrm{d}t & 0 & -F'\,\mathrm{d}r\wedge\mathrm{d}\theta & -F'\sin\theta\,\mathrm{d}r\wedge\mathrm{d}\phi - F\cos\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi \\ 0 & F'\,\mathrm{d}r\wedge\mathrm{d}\theta & 0 & \sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi \\ 0 & F'\sin\theta\,\mathrm{d}r\wedge\mathrm{d}\phi + F\cos\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi & -\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi & 0 \end{pmatrix} \tag{1.9}$$

and the wedge product

$$(\omega\wedge\omega)_\alpha{}^\beta = \begin{pmatrix} 0 & 0 & F^2N'\,\mathrm{d}t\wedge\mathrm{d}\theta & F^2N'\,\sin\theta\,\mathrm{d}t\wedge\mathrm{d}\phi \\ 0 & 0 & 0 & F\cos\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi \\ -F^2N'\,\mathrm{d}\theta\wedge\mathrm{d}t & 0 & 0 & -F^2\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi \\ -F^2N'\,\sin\theta\,\mathrm{d}\phi\wedge\mathrm{d}t & F\cos\theta\,\mathrm{d}\phi\wedge\mathrm{d}\theta & -F^2\sin\theta\,\mathrm{d}\phi\wedge\mathrm{d}\theta & 0 \end{pmatrix} \tag{1.10}$$

of connection matrices. The sum of these determines the Riemann matrix $R_{ab\alpha}{}^\beta$ of two-forms to be

$$R_\alpha{}^\beta = \begin{pmatrix} 0 & (FN')'\,\mathrm{d}t\wedge\mathrm{d}r & F^2N'\,\mathrm{d}t\wedge\mathrm{d}\theta & F^2N'\,\sin\theta\,\mathrm{d}t\wedge\mathrm{d}\phi \\ (FN')'\,\mathrm{d}t\wedge\mathrm{d}r & 0 & -F'\,\mathrm{d}r\wedge\mathrm{d}\theta & -F'\sin\theta\,\mathrm{d}r\wedge\mathrm{d}\phi \\ F^2N'\,\mathrm{d}t\wedge\mathrm{d}\theta & F'\,\mathrm{d}r\wedge\mathrm{d}\theta & 0 & (1-F^2)\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi \\ F^2N'\,\sin\theta\,\mathrm{d}t\wedge\mathrm{d}\phi & F'\sin\theta\,\mathrm{d}r\wedge\mathrm{d}\phi & -(1-F^2)\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi & 0 \end{pmatrix}. \tag{1.11}$$

Our real objective, of course, is to find the Einstein tensor, which entails calculating the Ricci tensor. To do this, we define the column vector

$$R_{a\alpha} := R_{ab\alpha}{}^\beta\, e^b_\beta \tag{1.12}$$

of one-forms. One can easily show that this object equals the Ricci tensor $R_{ab}\, e^b_\alpha$, with one index made concrete using the orthonormal basis. However, it is also easy to calculate using our matrix results above. We right-multiply the Riemann matrix (1.11) by the column vector (1.3) of basis vectors, while contracting those vectors with the *second* factor in the wedge products in the Riemann matrix. Minding the anti-symmetry of those wedge products, we get

$$R_\alpha = \begin{pmatrix} (FN')'\,F\,\mathrm{d}t + F^2N'\,r^{-1}\,\mathrm{d}t + F^2N'\,r^{-1}\,\mathrm{d}t \\ -(FN')'\,N^{-1}\,\mathrm{d}r - F'\,r^{-1}\mathrm{d}r - F'\,r^{-1}\,\mathrm{d}r \\ -F^2N'\,N^{-1}\mathrm{d}\theta - F'\,F\,\mathrm{d}\theta + (1-F^2)\,r^{-1}\,\mathrm{d}\theta \\ -F^2N'\,\sin\theta\,N^{-1}\,\mathrm{d}\phi - F'\sin\theta\,F\,\mathrm{d}\phi + (1-F^2)\sin\theta\,r^{-1}\mathrm{d}\phi \end{pmatrix}. \tag{1.13}$$

It is convenient now to collect terms and write the entries of this vector in terms of the orthonormal basis forms in (1.2). The result is

$$R_\alpha = \begin{pmatrix} FN^{-1}\,r^{-1}\,[r(FN')' + 2FN']\,e^t \\ -FN^{-1}\,r^{-1}\,[r(FN')' + 2NF']\,e^r \\ -r^{-2}\,[rF^2N'N^{-1} + rFF' - (1-F^2)]\,e^\theta \\ -r^{-2}\,[rF^2N'N^{-1} + rFF' - (1-F^2)]\,e^\phi \end{pmatrix}. \tag{1.14}$$

This Ricci tensor is diagonal in the sense that the one-form R_t is proportional to e^t and so forth. This should not be surprising since the symmetries of spacetime should constrain the components of its curvature in much the same way that they do those of the metric in (1.1). Also note that the coefficients multplying the angular basis vectors in the last two entries are identical, in accord with spherical symmetry.

The symmetries of the Ricci tensor emerge even more clearly when we write it out in full abstract index notation:

$$\begin{aligned} R_{ab} = R_{a\alpha}\,e^\alpha_b = \frac{F}{Nr}\left(r\big(FN'\big)' + 2FN'\right)e^t_a\,e^t_b - \frac{F}{Nr}\left(r\big(FN'\big)' + 2NF'\right)e^r_a\,e^r_b \\ - \frac{F}{Nr}\left(\big(FN\big)' - \frac{N}{Fr}(1-F^2)\right)\left(e^\theta_a\,e^\theta_b + e^\phi_a\,e^\phi_b\right). \end{aligned} \tag{1.15}$$

This expression allows us to calculate the scalar curvature

$$R = -\frac{2F}{Nr}\left(r\big(FN'\big)' + 2\big(FN\big)' - \frac{N}{Fr}\big(1-F^2\big)\right) \tag{1.16}$$

immediately, and thus to write down the Einstein tensor

$$G_{ab} := R_{ab} - \tfrac{1}{2}\,R\left(-e^t_a\,e^t_b + e^r_a\,e^r_b + e^\theta_a\,e^\theta_b + e^\phi_a\,e^\phi_b\right). \tag{1.17}$$

Collecting terms, we find the the Einstein tensor has the form

$$\begin{aligned} G_{ab} = \frac{1}{r^2}\left(r\left(1-F^2\right)\right)'e^t_a\,e^t_b + \frac{1}{r^2}\left(F^2r\left(\ln N^2\right)' - \left(1-F^2\right)\right)e^r_a\,e^r_b \\ + \frac{F}{Nr}\left(r\big(FN'\big)' + \big(FN\big)'\right)\left(e^\theta_a\,e^\theta_b + e^\phi_a\,e^\phi_b\right). \end{aligned} \tag{1.18}$$

This holds for any static, spherically symmetric metric (1.1).

Exercise 1.19: Starting from (1.15), collect terms to show that the Einstein tensor of a static, spherically symmetric spacetime can indeed be written in the form (1.18).

1.2 THE SCHWARZSCHILD SOLUTION

The Schwarzschild metric is a static, spherically symmetric solution of the *vacuum* Einstein equations. It is very nearly unique, having only one free constant of integration, denoted M. Physically, this represents the overall mass of the static, spherically symmetric body generating the gravitational field.

When we set $G_{ab} = 0$ in (1.18), we discover *three* ordinary differential equations for the *two* unknown metric component functions, $N(r)$ and $F(r)$. This seems a little odd, and we shall return to this conceptual difficulty below. For now, let's just start solving the equations. First, we have

$$0 = G_{tt} = \frac{1}{r^2}\Big(r\left(1 - F^2\right)\Big)' \quad \Rightarrow \quad F^2(r) = 1 - \frac{2M}{r}, \tag{1.20}$$

for some constant M. We have inserted the factor of two in order to recover the correct Newtonian limit when dimensionless parameter $M/r = GM/c^2r$ is small. To solve for $N(r)$, it is actually easiest to go back to the expression (1.15) for the Ricci tensor. Since vanishing Einstein tensor implies vanishing Ricci tensor, we have

$$0 = R_{tt} + R_{rr} = \frac{F}{Nr}\Big(2FN' - 2NF'\Big) \quad \Rightarrow \quad \frac{N'}{N} = \frac{F'}{F}. \tag{1.21}$$

Thus, we find that $N(r)$ is proportional to $F(r)$, whence

$$N^2(r) = C^2\left(1 - \frac{2M}{r}\right) \tag{1.22}$$

for some constant C. Examining the metric (1.1), we notice that we can absorb this constant into the definition of the static time coordinate t, setting $t \mapsto \tilde{t} := Ct$. The vector field $\partial_{\tilde{t}} = C^{-1}\,\partial_t$ still Lie drags the metric, which then has the form

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right), \tag{1.23}$$

where we have dropped the tilde on the rescaled time coordinate. This is the ***Schwarzschild metric***.