Physics 6938 Florida Atlantic University
General Relativity Fall, 2007

Problem Set V

Due: Friday, 7 December 2007
(A date which will live in infamy!)

1. Recall that the extrinsic curvature of a two-dimensional submanifold S of a three-dimensional Riemannian manifold (Σ, q_{ab}) was defined in class to be

$$\kappa^{ab} := -\sigma^{am}\,\nabla_m\,\hat{r}^b,$$

where $\hat{r}^b$ is the normal to S within Σ and σ^{am} is the two-dimensional metric on S. Let Σ be ordinary Euclidean space, and S a round sphere of radius a. Show that

$$\kappa^{ab} = -a^{-1}\,\sigma^{ab}.$$

Hint: Evaluate $\nabla_a\,\nabla_b\,r^2$ in (a) spherical and (b) Cartesian coordinates.

2. Consider the linearized plane gravitational wave $h_{ab}(\mathbf{x}) = A_{ab}\,e^{i\mathbf{k}\cdot\mathbf{x}}$ on Minkowski spacetime. Assume that it is in de Donder gauge, so that $k^a\,A_{ab} = 0$, but not necessarily in a transverse-traceless gauge.

   a. Let $\hat{u}^a$ be a given time-like unit vector, and define the spatial unit vector $\hat{k}^a$ by

$$k^a =: \omega\big(\hat{u}^a + \hat{k}^a\big) \quad \text{and} \quad \hat{u}^a\,\hat{k}_a = 0.$$

   In words, $\hat{k}^a$ is the unit vector along the direction of propagation for $h_{ab}(\mathbf{x})$ within the spatial slice orthogonal to $\hat{u}^a$. Define the projection operator

$$P_{ab} := \eta_{ab} + \hat{u}_a\,\hat{u}_b - \hat{k}_a\,\hat{k}_b$$

   into the 2-plane orthogonal to both $\hat{u}^a$ and $\hat{k}^a$. Show that

$$h^{\mathsf{TT}}_{ab}(\mathbf{x}) := A^{\mathsf{TT}}_{ab}\,e^{i\mathbf{k}\cdot\mathbf{x}} \quad \text{with} \quad A^{\mathsf{TT}}_{ab} := \Big(P_a{}^m\,P_b{}^n - \tfrac{1}{2}\,P_{ab}\,P^{mn}\Big)\,A_{mn}$$

   is a plane-wave solution of the homogeneous wave equation of linearized gravity, and that it obeys the complete set of transverse-traceless gauge conditions in the frame defined by $\hat{u}^a$, including the de Donder condition, as defined in class. As the notation suggests, this operation projects out the transverse-traceless part of a plane wave initially given in an arbitrary (but de Donder) gauge.

   b. Show that the transverse-traceless part of the wave amplitude may also be calculated from the formula

$$A^{\mathsf{TT}}_{ab} = \Big[\bar{\eta}^i_a\,\bar{\eta}^j_b - 2\,\hat{k}^{(i}\,\bar{\eta}^{j)}_{(a}\,\hat{k}_{b)} + \tfrac{1}{2}\big(\bar{\eta}_{ab} + \hat{k}_a\,\hat{k}_b\big)\,\hat{k}^i\,\hat{k}^j\Big]\Big[\bar{\eta}^m_i\,\bar{\eta}^n_j - \tfrac{1}{3}\,\bar{\eta}_{ij}\,\bar{\eta}^{mn}\Big]A_{mn},$$

   where $\bar{\eta}_{ab} := \eta_{ab} + \hat{u}_a\,\hat{u}_b$ denotes the spatial metric. Note that the first operator to act on A_{mn} yields the traceless part of its spatial projection.

3. Define the right- and left-handed circular polarization tensors by

$$e^{\circlearrowleft}_{ab} := \tfrac{1}{\sqrt{2}}\left(e^{+}_{ab} + \mathrm{i}\, e^{\times}_{ab}\right) \quad \text{and} \quad e^{\circlearrowright}_{ab} := \tfrac{1}{\sqrt{2}}\left(e^{+}_{ab} - \mathrm{i}\, e^{\times}_{ab}\right),$$

respectively. Consider a right-circularly-polarized gravitational plane wave with amplitude A propagating along the $+z$-direction. It is incident on a pair of non-accelerating particles separated by a distance d along the x-axis in the Minkowski background. Show that each particle moves in a right-handed circle relative to the other, as seen from above. What is the radius of that circle? What changes for a left-circularly-polarized wave? *Hint*: Recall that one must take the real part of the complex expression for the wave.

4. Consider a pair of identical point particles of mass m orbiting one another with frequency ω in a circle of radius a about the origin in the xy-plane.

   a. Write the energy density of the source in terms of delta functions and show that the resulting trace-reversed metric perturbation field is

   $$h_{ab}(t, \vec{r}) = \frac{8m}{r}\, u_a\, u_b - \frac{8ma^2\omega^2}{r}\left[\cos\!\big(2\omega(t-r)\big)\, e^{+z}_{ab} + \sin\!\big(2\omega(t-r)\big)\, e^{\times z}_{ab}\right].$$

   Here, e^{+z}_{ab} and $e^{\times z}_{ab}$ are the "plus" and "cross" polarization tensors defined in class for waves propagating in the $+z$-direction.

   b. At large radius, the spherical waves produced by the source are very nearly planar, and we can extract their transverse traceless parts by assuming that the direction $\hat{k}^a$ of propagation in the problems above is identical to the direction $\hat{r}^a$ from the origin to the observation point. Show that an observer on the $+z$-axis will observe the wave

   $$h^{\mathsf{TT}}_{ab}(t, r\hat{z}) = -\frac{8\sqrt{2}ma^2\omega^2}{r}\,\mathrm{Re}\!\left[\mathrm{e}^{-2\mathrm{i}\omega(t-r)}\, e^{\circlearrowleft z}_{ab}\right].$$

   That is, an observer above the source will see right-circularly polarized waves.

   c. In contrast, show that an observer at large distance along the $+x$-axis will measure linearly polarized waves with

   $$h^{\mathsf{TT}}_{ab}(t, r\hat{x}) = \frac{4ma^2\omega^2}{r}\,\mathrm{Re}\!\left[\mathrm{e}^{-2\mathrm{i}\omega(t-r)}\, e^{+x}_{ab}\right],$$

   where $e^{+x}_{ab} := \hat{e}^{y}_{a}\,\hat{e}^{y}_{b} - \hat{e}^{z}_{a}\,\hat{e}^{z}_{b}$ defines the "plus" polarization along the $+x$-axis.