Physics 6938 | Florida Atlantic University
General Relativity | Fall, 2007

Problem Set II

Due: Thursday, 11 October 2007

1. [based on *d'Inverno* 8.3] Suppose the vectors T^a, X^a, Y^a and Z^a form an orthonormal tetrad for Minkowski spacetime, so that the only non-vanishing inner products between them are $T^a\,T_a = -1$ and $X^a\,X_a = Y^a\,Y_a = Z^a\,Z_a = 1$. Define the vectors

$$L^a = \frac{T^a + Z^a}{\sqrt{2}} \qquad N^a = \frac{T^a - Z^a}{\sqrt{2}}$$
$$M^a = \frac{X^a + \mathrm{i}Y^a}{\sqrt{2}} \qquad \bar{M}^a = \frac{X^a - \mathrm{i}Y^a}{\sqrt{2}}.$$

   Show that all of these vectors are null in the real inner product defined by the Minkowski metric η_{ab}, and that their only non-vanishing inner products are $L^a\,N_a = -1$ and $M^a\,\bar{M}_a = 1$.

2. [based on *d'Inverno* 8.5] Show that a Killing vector X_c satisfies $\nabla_a\,\nabla_b\,X_c = 0$ in Minkowski spacetime, where ∇_a represents the flat metric connection. Deduce from this that the vector field X_a can be written in the form

$$X_a(x) = \omega_{ab}\,x^b + t_a,$$

   where $\omega_{ab} = \omega_{[ab]}$ and t_a are constant parameters, and x^b is the vector pointing from the origin to the event where the vector field is evaluated. How many parameters determine X_a in (a) an n-dimensional manifold and (b) Minkowski spacetime. What do these parameters correspond to physically in the latter case?

3. [*d'Inverno* 9.2] Consider a body rotating relative to an inertial frame about a fixed point O with angular velocity $\boldsymbol{\omega}$ *in Newtonian theory*. The velocity $\mathbf{v}$ of the point P in the body with position $\mathbf{r} = OP$ is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

   Let $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denote unit vectors in the inertial frame S and $\mathbf{i}'$, $\mathbf{j}'$ and $\mathbf{k}'$ denote unit vectors in a frame S' fixed to the body, where both origins are at O. If $\mathbf{u} = \mathbf{u}(t)$ is a general vector with components

$$\mathbf{u} = u'_a\,\mathbf{i}' + u'_2\,\mathbf{j}' + u'_3\,\mathbf{k}'$$

   in S', show, by differentiating this equation, that

$$\left[\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right]_S = \left[\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\right]_{S'} + \boldsymbol{\omega} \times \mathbf{u}.$$

4. [*d'Inverno* 9.2] Consider a non-inertial frame S' moving arbitrarily relative to an inertial frame S *in Newtonian theory*, where the position of the origin O' of S' relative to the origin O of S is $\mathbf{s}(t)$ and the angular velocity of S' relative to S is $\boldsymbol{\omega}(t)$. A particle of constant mass m situated at a point with position vectors $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ relative to S and S', respectively, is acted on by a forces $\mathbf{F}$. Show that S' can write the equation of motion of the particle in the form

$$m\ddot{\mathbf{r}}' = \mathbf{F} - m\left[\mathbf{a} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}}' + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}') + \dot{\boldsymbol{\omega}} \times \mathbf{r}'\right],$$

where $\mathbf{a}$ is the acceleration of O' relative to O and a dot denotes differentiation with respect to time in the frame of S'. What are the quantities in square brackets? Interpret these quantities physically.

5. [*d'Inverno* 9.10] An anti-symmetric tensor F_{ab} satisfies the special-relativistic homogeneous Maxwell equation

$$\partial_{[a} F_{bc]} = 0,$$

where ∂_a is the coordinate connection in inertial coordinates on Minkowski spacetime. Write down the simplest generalization of this to a curved spacetime and show that it is identical to the original equation in *all* coordinate systems.