Geometry Exercises V

1 The metric is

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\rightsquigarrow \quad 2K = e^\nu \dot{t}^2 - e^\lambda \dot{r}^2 - r^2(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2)$$

This gives the geodesic equations

$$e^\nu \frac{\partial\nu}{\partial t}\dot{t}^2 - e^\lambda \frac{\partial\lambda}{\partial t}\dot{r}^2 - \frac{d}{du}(2e^\nu \dot{t}) = 0$$

$$e^\nu \frac{\partial\nu}{\partial r}\dot{t}^2 - e^\lambda \frac{\partial\lambda}{\partial r}\dot{r}^2 - 2r(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) - \frac{d}{du}(-2e^\lambda \dot{r}) = 0$$

$$-2r^2 \sin\theta\cos\theta\, \dot{\phi}^2 - \frac{d}{du}(-2r^2\dot{\theta}) = 0$$

$$-\frac{d}{du}(-2r^2\sin^2\theta\, \dot{\phi}) = 0$$

We need to expand these derivatives:

$$2e^\nu \ddot{t} + e^\nu \frac{\partial\nu}{\partial t}\dot{t}^2 + 2e^\nu \frac{\partial\nu}{\partial r}\dot{r}\dot{t} + e^\lambda \frac{\partial\lambda}{\partial t}\dot{r}^2 = 0$$

$$2e^\lambda \ddot{r} + e^\nu \frac{\partial\nu}{\partial r}\dot{t}^2 + 2e^\lambda \frac{\partial\lambda}{\partial t}\dot{t}\dot{r} + e^\lambda \frac{\partial\lambda}{\partial r}\dot{r}^2 - 2r(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) = 0$$

$$2r^2\ddot{\theta} + 4r\dot{r}\dot{\theta} - 2r^2\sin\theta\cos\theta\, \dot{\phi}^2 = 0$$

$$2r^2\sin^2\theta\, \ddot{\phi} + 4r\sin^2\theta\, \dot{r}\dot{\phi} + 4r^2\sin\theta\cos\theta\, \dot{\theta}\dot{\phi} = 0$$

Now we need to divide through by the coefficients of the second derivatives.

2 The results are

$$\ddot{t} + \frac{1}{2}\frac{\partial \nu}{\partial t}\dot{t}^2 + \frac{\partial \nu}{\partial r}\dot{t}\dot{r} + \frac{1}{2}e^{(\lambda-\nu)}\frac{\partial \lambda}{\partial t}\dot{r}^2 = 0$$

$$\ddot{r} + \frac{1}{2}e^{(\nu-\lambda)}\frac{\partial \nu}{\partial r}\dot{t}^2 + \frac{\partial \lambda}{\partial t}\dot{t}\dot{r} + \frac{1}{2}\frac{\partial \lambda}{\partial r}\dot{r}^2 - re^{-\lambda}\dot{\theta}^2 - re^{-\lambda}\sin^2\theta\,\dot{\phi}^2 = 0$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\,\dot{\phi}^2 = 0$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\,\dot{\theta}\dot{\phi} = 0$$

This lets us read off the Christoffel components from (7.42):

$$\Gamma^t_{tt} = \frac{1}{2}\frac{\partial \nu}{\partial t} \qquad \Gamma^t_{tr} = \frac{1}{2}\frac{\partial \nu}{\partial r} \qquad \Gamma^t_{rr} = \frac{1}{2}e^{(\lambda-\nu)}\frac{\partial \lambda}{\partial t}$$

$$\Gamma^r_{tt} = \frac{1}{2}e^{(\nu-\lambda)}\frac{\partial \nu}{\partial r} \qquad \Gamma^r_{tr} = \frac{1}{2}\frac{\partial \lambda}{\partial t} \qquad \Gamma^r_{rr} = \frac{1}{2}\frac{\partial \lambda}{\partial r}$$

$$\Gamma^r_{\theta\theta} = -re^{-\lambda} \qquad \Gamma^r_{\phi\phi} = -re^{-\lambda}\sin^2\theta$$

$$\Gamma^\theta_{r\theta} = \frac{1}{r} \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$$

$$\Gamma^\phi_{r\phi} = \frac{1}{r} \qquad \Gamma^\phi_{\theta\phi} = \cot\theta$$

These agree precisely with the answers to problem 6.31(ii) in the book.

2 a) This is easy:

$$\mathcal{L}_U \mathcal{L}_V f - \mathcal{L}_V \mathcal{L}_U f = \mathcal{L}_U V(f) - \mathcal{L}_V U(f)$$
$$= U(V(f)) - V(U(f)) = [U,V](f) = \mathcal{L}_{[U,V]} f$$

b) For a vector field m^a, we find

$$\mathcal{L}_U \mathcal{L}_V m - \mathcal{L}_V \mathcal{L}_U m = \mathcal{L}_U [V, m] - \mathcal{L}_V [U, m]$$

$$= [U, [V, m]] - [V, [U, m]]$$

$$= [U, [V, m]] + [V, [m, U]] + [m, [U, V]] - [m, [U, V]]$$

$$= [[U, V], m] = \mathcal{L}_{[U,V]} m$$

We have used the Jacobi identity in the last line, so only the last term survives.

c) Here, we have

$$\mathcal{L}_U \mathcal{L}_V (m^a p_a) - \mathcal{L}_V \mathcal{L}_U (m^a p_a) =$$

$$= \mathcal{L}_U (p_a \mathcal{L}_V m^a + m^a \mathcal{L}_V p_a) - \mathcal{L}_V (p_a \mathcal{L}_U m^a + m^a \mathcal{L}_U p_a)$$

$$= p_a \mathcal{L}_U \mathcal{L}_V m^a + m^a \mathcal{L}_U \mathcal{L}_V p_a + \mathcal{L}_U p_a \cdot \mathcal{L}_V m^a + \mathcal{L}_U m^a \cdot \mathcal{L}_V p_a$$
$$- p_a \mathcal{L}_V \mathcal{L}_U m^a - m^a \mathcal{L}_V \mathcal{L}_U p_a - \mathcal{L}_V p_a \cdot \mathcal{L}_U m^a - \mathcal{L}_V m^a \cdot \mathcal{L}_U p_a$$

$$= p_a (\mathcal{L}_U \mathcal{L}_V m^a - \mathcal{L}_V \mathcal{L}_U m^a) + m^a (\mathcal{L}_U \mathcal{L}_V p_a - \mathcal{L}_V \mathcal{L}_U p_a)$$

$$= p_a \mathcal{L}_{[U,V]} m^a + m^a (\mathcal{L}_U \mathcal{L}_V p_a - \mathcal{L}_V \mathcal{L}_U p_a)$$

$$= \mathcal{L}_{[U,V]} (p_a m^a) + m^a (\mathcal{L}_U \mathcal{L}_V p_a - \mathcal{L}_V \mathcal{L}_U p_a - \mathcal{L}_{[U,V]} p_a)$$

The first term on the right is equal to the left side by part (a). The result follows since m^a is arbitrary.

4

d) This is immediate

$$\mathcal{L}_{[U,V]}\, g_{ab} = \mathcal{L}_U \mathcal{L}_V g_{ab} - \mathcal{L}_V \mathcal{L}_U g_{ab}$$

Both terms on the right vanish when U and V are both Killing fields, whence $[U,V]$ is a Killing field as well

e) Using the previous result, we calculate

$$[\partial_x, -y\partial_x + x\partial_y] = \partial_y$$

is a Killing field.

3 a) We calculate

$$\nabla_a \nabla_b X_c = -\nabla_a \nabla_c X_b = -R_{acb}{}^d X_d - \nabla_c \nabla_a X_b$$

$$= R_{cab}{}^d X_d - \nabla_c \nabla_a X_b$$

$$= R_{cab}{}^d X_d - (R_{bca}{}^d X_d - \nabla_b \nabla_c X_a)$$

$$= (R_{cab}{}^d - R_{bca}{}^d) X_d + (R_{abc}{}^d X_d - \nabla_a \nabla_b X_c)$$

$$\Rightarrow 2\nabla_a \nabla_b X_c = (R_{abc}{}^d - R_{bca}{}^d + R_{cab}{}^d) X_d$$

$$= -2R_{bca}{}^d X_d = 2R_{cba}{}^d X_d$$

We have used the Bianchi identity in the final line. The result follows.

b) We now use the metric g^{ab} to contract the previous result:

$$\nabla^b \nabla_b X_c = R_{cb}{}^{bd} X_d = -R_c{}^d X_d$$

The sign difference arises from the different conventions surrounding the definition of the curvature.

4 Here, we calculate

$$X^a Y^b \mathcal{L}_V g_{ab} = \mathcal{L}_V (X^a Y^b g_{ab}) - X_b \mathcal{L}_V Y^b - Y_a \mathcal{L}_V X^a$$

$$= V^m \nabla_m (X^a Y^b g_{ab}) - X_b (V^m \nabla_m Y^b - Y^m \nabla_m V^b$$

$$- Y_a (V^m \nabla_m X^a - X^m \nabla_m V^a)$$

$$= g_{ab} V^m (X^a \nabla_m Y^b + Y^b \nabla_m X^a)$$

$$- X_b V^m \nabla_m Y^b + X^b Y^m \nabla_m V_b - Y_a V^m \nabla_m X^a + X^m Y^a \nabla_m V_a$$

$$= X^a Y^b \nabla_b V_a + X^a Y^b \nabla_a V_b = 2 X^a Y^b \nabla_{(a} V_{b)}$$

The result follows because X^a and Y^b are both arbitrary.

5 Let's do both parts at once here. Suppose that X^a_α are a collection of vector fields such that $[X_\alpha, X_\beta] = c_{\alpha\beta}{}^\gamma X_\gamma$, and that $V^a = f^\alpha X^a_\alpha$ and $W^a = g^\alpha X^a_\alpha$.

6 We now use the result

$$[Y, fZ] = Y(f)\,Z + f[Y, Z]$$

to write

$$\begin{aligned}[U, V] &= [f^\alpha X_\alpha,\ g^\beta X_\beta] \\ &= f^\alpha X_\alpha(g^\beta) X_\beta + g^\beta [f^\alpha X_\alpha, X_\beta] \\ &= f^\alpha X_\alpha(g^\beta) X_\beta - g^\beta X_\beta(f^\alpha) X_\alpha + f^\alpha g^\beta [X_\alpha, X_\beta] \\ &= [f^\alpha X_\alpha(g^\gamma) - g^\beta X_\beta(f^\gamma) + f^\alpha g^\beta c_{\alpha\beta}{}^\gamma] X_\gamma\end{aligned}$$

Thus, $[U, V]$ can be written as a (functional) linear combination of the X_γ.

6 Here, we write

$$\begin{aligned}\nabla_\eta(g_{ab}\,\eta^a \xi^b) &= g_{ab}(\xi^b \nabla_\eta \eta^a + \eta^a \nabla_\eta \xi^b) \\ &= \eta^a \eta^m \nabla_m \xi_a = \eta^a \eta^m \nabla_{(m}\xi_{a)} = 0\end{aligned}$$

Here, we have used the affine geodesic equation $\nabla_\eta \eta^a = 0$ and the Killing equation $\nabla_{(m}\xi_{a)} = 0$.

If we change affine parameterizations, η^a scales by a constant, and so $\eta \cdot \xi$ does as well.

If the parameterization is not affine, then $\nabla_\eta \eta^a = \alpha \eta^a$, with α a non-zero function along the geodesic. In this case, we find

$$\nabla_\eta (\eta \cdot \xi) = \xi \cdot \nabla_\eta \eta = \alpha\, \xi \cdot \eta$$

by the calculation above. The product is no longer constant, but its rate of change is given by the "temporal acceleration" α of the geodesic.

7 a) The homogeneous Maxwell equation follows from the Bianchi identity:

$$\nabla_{[a} F_{bc]} = \nabla_{[a} \nabla_b t_{c]} = \tfrac{1}{2} R_{[abc]}{}^{d}\, t_d = 0$$

The inhomogeneous Maxwell equation follows from exercise 3b above:

$$\nabla_a F^{ab} = \nabla_a \nabla^a t^b = -R^{ab}\, t_a = 0$$

because $G_{ab} = 0$ implies $R_{ab} = 0$:

$$g^{ab} G_{ab} = R - \tfrac{1}{2} R\, \delta^a_a = R - \tfrac{1}{2} \cdot 4 R = -R$$

$$\therefore \; G_{ab} = 0 \;\Rightarrow\; R = 0 \;\Rightarrow\; R_{ab} = \tfrac{1}{2} R g_{ab} = 0$$

b) Hyper-surface orthogonality of t^a implies

$$t_{[a} \nabla_b t_{c]} = 0 \;\Rightarrow\; F_{bc} = 2\, t_{[b}\, C_{c]}$$

8

for some 1-form c_c. Generally, a Maxwell tensor can be written

$$F_{ab} = 2 n_{[a} E_{b]} + B_{ab}$$

where $E_a = F_{ab} n^b$ is the electric field 1-form and B_{ab} is the magnetic field 2-form, and n_a is the normal to a hypersurface Σ on which the fields are measured. It follows immediately that

$$E_b = \|t\| c_b,$$

and our F_{ab} in this problem is pure electric on the static slices Σ_t.

c) Here, we calculate

$$t_a = g_{ab} \left(\frac{\partial}{\partial t}\right)^b = -\left(1 - \frac{2M}{r}\right) \nabla_a t$$

$$\Rightarrow F_{ab} = \nabla_a t_b = \nabla_{[a} t_{b]} = -\nabla_{[a}\left(1 - \frac{2M}{r}\right) \cdot \nabla_{b]} t$$

$$\Rightarrow E_a = F_{ab} n^b = F_{ab} \frac{t^b}{\sqrt{1-\frac{2M}{r}}} = -\frac{1}{2} \nabla_a \left(1 - \frac{2M}{r}\right) \cdot \frac{1}{\sqrt{1-\frac{2M}{r}}}$$

$$= -\nabla_a \sqrt{1 - \frac{2M}{r}} = -\nabla_a \left(1 - \frac{M}{r} + \frac{M^2}{2r^2} - \cdots\right)$$

$$= -\frac{M}{r^2} \nabla_a r + \frac{M^2}{r^3} \nabla_a r$$

If we take a flux integral over a very large sphere, only the first term survives, and we find the "charge" of the Schwarzschild metric is $Q = -M$.