Geometry Exercises I

1 The relations are

$$r = \sqrt{x^2+y^2+z^2}, \quad \theta = \tan^{-1}\frac{\sqrt{x^2+y^2}}{z}, \quad \phi = \tan^{-1}\frac{y}{x}$$

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta.$$

From the second set of expressions, we can easily calculate the matrix

$$\frac{\partial x^\alpha}{\partial x'^\beta} = \begin{array}{cc} \begin{pmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix} & \begin{matrix} \leftarrow x \\ \leftarrow y \\ \leftarrow z \end{matrix} \\ \begin{matrix} \uparrow \qquad\qquad & \uparrow \qquad\qquad & \uparrow \\ r \qquad\qquad & \theta \qquad\qquad & \phi \end{matrix} & \end{array}$$

We use the first set of expressions temporarily to calculate the inverse

$$\frac{\partial x'^\beta}{\partial x^\sigma} = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} & \frac{z}{r} \\ \frac{xz}{r^2\sqrt{x^2+y^2}} & \frac{yz}{r^2\sqrt{x^2+y^2}} & -\frac{\sqrt{x^2+y^2}}{r^2} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{pmatrix}$$

$$= \begin{array}{cc} \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{1}{r}\cos\theta\cos\phi & \frac{1}{r}\cos\theta\sin\phi & -\frac{1}{r}\sin\theta \\ \frac{-\sin\phi}{r\sin\theta} & \frac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix} & \begin{matrix} \leftarrow r \\ \leftarrow \theta \\ \leftarrow \phi \end{matrix} \\ \begin{matrix} \uparrow \qquad\qquad & \uparrow \qquad\qquad & \uparrow \\ x \qquad\qquad & y \qquad\qquad & z \end{matrix} & \end{array}$$

The determinants of these are

$$J = \cos\theta\left(r^2\cos\theta\sin\theta\right) + r\sin\theta\left(r\sin^2\theta\right) = r^2\sin\theta$$

$$J' = \cos\theta\left(\frac{\cos\theta}{r^2\sin\theta}\right) + \frac{1}{r}\sin\theta\left(\frac{1}{r}\right) = \frac{1}{r^2\sin\theta}$$

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J' is infinite along the z-axis, where $\sin\theta = 0$. This includes the origin, where $r = 0$.

2 Here, the transformation is

$$r = \sqrt{x^2+y^2}, \qquad \theta = \tan^{-1}\frac{y}{x}$$

$$x = r\cos\theta, \qquad y = r\sin\theta$$

We have

$$\begin{array}{cc} \dfrac{\partial x'}{\partial x} = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix} & \begin{array}{l} \leftarrow r \\ \leftarrow \theta \end{array} \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \begin{array}{cc} \uparrow & \uparrow \\ x & y \end{array} & \end{array}$$

The standard parameter is the arc length $s = a\theta$. The tangent vector is

$$V = \frac{\partial}{\partial s} = \frac{1}{a}\frac{\partial}{\partial\theta} = \frac{1}{a}\left(\frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y}\right)$$

$$= \frac{1}{a}\left(-r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}\right)_{r=a}$$

$$= -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}$$

This is just the unit vector $\hat{\phi}$ in conventional notation.

3 The derivative is

$$\frac{\partial^2\phi}{\partial x'^\alpha\partial x'^\beta} = \frac{\partial^2 x^\gamma}{\partial x'^\alpha\,\partial x'^\beta}\frac{\partial\phi}{\partial x^\gamma} + \frac{\partial x^\gamma}{\partial x'^\alpha}\cdot\frac{\partial x^\delta}{\partial x'^\beta}\frac{\partial}{\partial x^\delta}\frac{\partial\phi}{\partial x^\gamma}$$

We have used the chain rule in the second term, which has the correct transformation rule for a tensor. But the first term does not, so second partial derivatives are not tensors.

4 a) Suppose that $T_{\alpha\beta} = T_{\beta\alpha}$. Then

$$T_{\alpha'\beta'} = \frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}} T_{\alpha\beta} = \frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^\beta}{\partial x^{\beta'}} T_{\beta\alpha}$$

$$= \frac{\partial x^\delta}{\partial x^{\alpha'}}\frac{\partial x^\gamma}{\partial x^{\beta'}} T_{\gamma\delta} = T_{\beta'\alpha'}$$

b) Here, we write

$$X^{ab} Y_{ab} = X^{ab} Y_{ba} = -X^{ba} Y_{ba} = -X^{cd} X_{cd} = -X^{ab} X_{ab}$$

5 Here, we pick an arbitrary coordinate system:

$$\delta^a_a = \delta^\alpha_\alpha = \sum_{\alpha=1}^{n} 1 = n$$

For the square, we have $\delta^a_b \delta^b_a := \delta^a_a = n$.

6 The first two are obvious:

$$[X, X] = XX - XX = 0$$

$$[X, Y] = XY - YX = -(YX - XY) = -[Y, X]$$

4 For the Jacobi identity, we write

$$[X, [Y, Z]] = [X, YZ - ZY] = X(YZ - ZY) - (YZ - ZY)X$$

$$= XYZ - XZY - YZX + ZYX$$

$$= (XYZ - YZX) + (ZYX - XZY)$$

In this last expression, we have collected the two even permutations of XYZ in the first term, and the two odd permutations in the second. When we sum over the cyclic permutations on the left, each permutation on the right occurs twice, once with each sign. The Jacobi identity follows immediately

7 a) We can read the components off from

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}$$

$$= \frac{x}{r}\frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial \theta} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

b) Likewise, we have

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y}$$

7 c) We have already done this in (a).

d) From part b, we have

$$Y = \frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

$$Z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = -r\sin\theta\frac{\partial}{\partial x} + r\cos\theta\frac{\partial}{\partial y} = \frac{\partial}{\partial\theta}$$

e) We have

$$[X, Y] = \frac{\partial}{\partial x}\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\frac{\partial}{\partial x} = 0$$

$$[X, Z] = \frac{\partial}{\partial x}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) - \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$$

$$[Y, Z] = \frac{\partial}{\partial y}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) - \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\frac{\partial}{\partial y} = -\frac{\partial}{\partial x}$$

8 a) A general $n \times n$ matrix has n^2 components, while two vectors (and therefore their outer product) have $2n$.

b) The components are given by

$$(\bar{V} \otimes \tilde{\omega})^i{}_j := \bar{V} \otimes \tilde{\omega}(\tilde{e}^i, \bar{e}_j)$$

$$:= \bar{V}(\tilde{e}^i)\,\tilde{\omega}(\bar{e}_j) := V^i w_j$$

c) We need to define the linear combination of two tensors. The definition is obvious:

$$(\alpha T + \alpha' T')(\tilde{\omega}_1, \tilde{\omega}_2) := \alpha T(\tilde{\omega}_1, \tilde{\omega}_2) + \alpha' T'(\tilde{\omega}_1, \tilde{\omega}_2)$$

But we also need to show that the result is a tensor, that it is linear in its arguments:

$$(\alpha T + \alpha' T')(\beta \tilde{\omega}_1 + \beta' \tilde{\omega}_1', \tilde{\omega}_2) =$$

$$= \alpha T(\beta \tilde{\omega}_1 + \beta' \tilde{\omega}_1', \tilde{\omega}_2) + \alpha' T'(\beta \tilde{\omega}_1 + \beta' \tilde{\omega}_1', \tilde{\omega}_2)$$

$$= \alpha [\beta T(\tilde{\omega}_1, \tilde{\omega}_2) + \beta' T(\tilde{\omega}_1', \tilde{\omega}_2)] + \alpha' [\beta T'(\tilde{\omega}_1, \tilde{\omega}_2) + \beta' T'(\tilde{\omega}_1', \tilde{\omega}_2)]$$

$$= \beta [\alpha T(\tilde{\omega}_1, \tilde{\omega}_2) + \alpha' T'(\tilde{\omega}_1, \tilde{\omega}_2)] + \beta' [\alpha T(\tilde{\omega}_1', \tilde{\omega}_2) + \alpha' T'(\tilde{\omega}_1', \tilde{\omega}_2)]$$

$$= \beta (\alpha T + \alpha' T')(\tilde{\omega}_1, \tilde{\omega}_2) + \beta' (\alpha T + \alpha' T')(\tilde{\omega}_1', \tilde{\omega}_2)$$

Thus, the linear combination of tensors is linear in its first argument. The proof for the second argument is identical.

To show that $\bar{e}_i \otimes \bar{e}_j$ is a basis, we write

$$T(\tilde{\omega}, \tilde{\eta}) = T(\omega^i \tilde{e}_i, \eta^j \tilde{e}_j) = \omega^i \eta^j T(\tilde{e}_i, \tilde{e}_j)$$

$$= \bar{e}^i(\tilde{\omega})\, \bar{e}^j(\tilde{\eta})\, T(\tilde{e}_i, \tilde{e}_j)$$

$$= T(\tilde{e}_i, \tilde{e}_j)\, \bar{e}^i \otimes \bar{e}^j(\tilde{\omega}, \tilde{\eta})$$

Thus, using the dual basis $\tilde{e}_i$, we can express every T as a linear combination of $\bar{e}^i \otimes \bar{e}^j$, which is exactly the definition of a basis

9. We need to show that $B \circ A$ is linear

$$B(A(\alpha \bar{V} + \alpha' \bar{V}')) = B(\alpha A(V) + \alpha' A(V'))$$

$$= \alpha B(A(V)) + \alpha' B(A(V'))$$

Thus, C is linear. Its components are

$$C^i{}_j = \tilde{e}^i \circ B(A(\bar{e}_j))$$

$$= \tilde{e}^i \circ B(\tilde{e}^k \circ A(\bar{e}_j) \cdot \bar{e}_k)$$

$$= \tilde{e}^k \circ A(\bar{e}_j) \cdot \tilde{e}^i \circ B(\bar{e}_k) = A^k{}_j \, B^i{}_k$$

This is basically identical to the proof of (1.30).

10 a) Let's view g^{-1} as a map from 1-forms to vectors. We need to show that it is linear. So consider

$$g^{-1}(\alpha \tilde{\omega} + \beta \tilde{\eta}) - \alpha g^{-1}(\tilde{\omega}) - \beta g^{-1}(\tilde{\eta}) =: \bar{V}$$

Apply g to this vector, and recall that (a) g is linear and (b) $g \circ g^{-1}$ is the identity:

$$g(\bar{V}) = \alpha \tilde{\omega} + \beta \tilde{\eta} - \alpha \tilde{\omega} - \beta \tilde{\eta} = 0$$

But g is non-degenerate, so $\bar{V} = 0$. Thus, g^{-1} must be linear.

b) First, if $g(\bar{e}_i, \bar{e}_j) = \pm \delta_{ij}$, then

$$[g(\bar{e}_i)](\bar{e}_j) = \pm \delta_{ij} \quad \Rightarrow \quad g(\bar{e}_i) = \pm \tilde{\omega}^i$$

That is, up to a sign, g maps the basis to the dual basis. Then,

$$\begin{aligned} g^{-1}(\tilde{\omega}^i, \tilde{\omega}^j) &= [g^{-1}(\tilde{\omega}^i)](\tilde{\omega}^j) \\ &= [g^{-1}(\pm g(\bar{e}_i))](\tilde{\omega}^j) \\ &= [\pm \bar{e}_i](\tilde{\omega}^j) = \pm \delta_i^j \end{aligned}$$

since the dual basis of the dual basis is the basis. This proves the result.