Lecture 1

Linearized Gravity

General relativity has several features that raise peculiar difficulties in applications of practical interest. These features include the non-linearity, over-determination and under-determination of the field equations, as well as the absence of a background geometry that might be used to define physically interesting quantities like momentum or energy. Due to these technical and conceptual difficulties, one must almost always make some approximations to make physically interesting predictions. This lecture deals with one such approximation scheme, linearized gravity.

Like the name suggests, linearized gravity is a linear field theory, meaning that its solutions obey the principle of superposition. It applies in situations where the gravitational field is dominated by a given solution of the Einstein equations that is known by some other means. Linearized gravity does allow additional sources in the problem, but these must be weak enough that they generate only a small correction in the overall field. Linearized gravity asks what condition is imposed on this perturbative correction to the field when we insist that the Einstein equations continue to hold *to first order in perturbation theory.*

The perturbation theory underlying linearized gravity must involve and expansion in some small parameter that we will denote λ. The physical meaning of this parameter may vary depending on the physical context for the approximation and need not be clarified in order to do the mathematical calculations below. Thus, we will forego a precise physical interpretation of λ in this discussion, and focus the general issues surrounding a perturbation expansion of the Einstein equation.

1.1 FIRST-ORDER GRAVITATIONAL PERTURBATION THEORY

We imagine that the parameter λ in which the perturbation expansion will occur may be tuned to take arbitrarily small values. For each, we suppose that we could find, in principle, a metric $g_{ab}(\lambda)$ on the spacetime manifold M that solves the Einstein equations. That is, for each λ, we suppose there is a metric

$$g_{ab}(\lambda) \quad \text{such that} \quad G_{ab}(\lambda) = 8\pi\, T_{ab}(\lambda). \tag{1.1}$$

Note that we allow the matter source to depend on λ as well since, in many applications, the source of the perturbation will include, for example, a small star moving through a region of spacetime. We also suppose that the ***background metric***

$$\mathring{g}_{ab} := g_{ab}(\lambda = 0) \tag{1.2}$$

on the manifold M is known exactly, and solves the Einstein equation

$$\mathring{G}_{ab} = 8\pi\, \mathring{T}_{ab} \tag{1.3}$$

for a given source tensor field $\mathring{T}_{ab}$.

We now imagine expanding the ***physical metric*** $g_{ab}(\lambda)$ as a power series in λ about $\lambda = 0$. The first-order coefficient in this expansion is given by

$$\dot{g}_{ab} := \left.\frac{\partial g_{ab}}{\partial \lambda}\right|_{\lambda=0} \tag{1.4}$$

at each point of spacetime. Now, when we modify the spacetime geometry by changing λ, we induce a change in the metric connection, and thus in the spacetime curvature. Symbolically, we have

$$g_{ab}(\lambda) \quad \rightsquigarrow \quad \nabla_a(\lambda) \quad \rightsquigarrow \quad R_{abc}{}^{d}(\lambda) \quad \rightsquigarrow \quad G_{ab}(\lambda). \tag{1.5}$$

A randomly chosen modification $g_{ab}(\lambda)$ of the metric will not generally yield an Einstein tensor $G_{ab}(\lambda)$ equal to the modified stress-energy tensor $T_{ab}(\lambda)$, which we assume is given. Therefore, in perturbation theory, we must ask what constraint is imposed on the first-order ***metric perturbation*** $\dot{g}_{ab}$ in the metric when we demand that the Einstein equation continue to hold at first order:

$$\left.\frac{\partial G_{ab}}{\partial \lambda}\right|_{\lambda=0} =: \dot{G}_{ab} = 8\pi\, \dot{T}_{ab} := 8\pi\, \left.\frac{\partial T_{ab}}{\partial \lambda}\right|_{\lambda=0}. \tag{1.6}$$

This constraint is the ***field equation of linearized gravity***. To find it, we must ask how the first-order change $\dot{g}_{ab}$ in the metric induces a first-order change $\dot{G}_{ab}$ in the Einstein tensor.

The Einstein tensor is determined by the Riemann curvature of the metric connection. Therefore, we must begin by sorting out the first-order correction to the connection. This correction is the difference between two connections, $\nabla_a(\epsilon)$ and $\nabla_a(0)$, in the limit where ϵ goes to zero, and therefore is given by a tensor that we denote $\dot{\nabla}_{ab}{}^{c}$. For each λ, we will have

$$\nabla_a(\lambda)\, g_{bc}(\lambda) = 0. \tag{1.7}$$

The derivative of this relation with respect to λ may act either on the derivative operator ∇_a—producing two terms, one for each covariant index of the metric tensor—or on the metric within. Since the right side does not depend on λ, we find

$$\dot{\nabla}_{ab}{}^{m}\, g_{mc} + \dot{\nabla}_{ac}{}^{m}\, g_{bm} + \nabla_a\, \dot{g}_{bc} = 0. \tag{1.8}$$

We have left the λ-dependence of this result implicit and have not set $\lambda = 0$ just yet, even though this is the only relevant value of that parameter in the present calculation. In principle, one could take a second derivative with respect to λ to move into second-order perturbation theory, though we will not do so here. This result can be reorganized to give one constraint

$$2\,\dot{\nabla}_{a(bc)} = -\nabla_a\,\dot{g}_{bc} \tag{1.9}$$

on the ***connection perturbation*** $\dot{\nabla}_{ab}{}^c$ in terms of the metric perturbation $\dot{g}_{ab}$. Note that we are using the background metric to lower the index on the connection perturbation here. A second constraint arises when we recall that ∇_a is meant to be torsion-free for all λ. Thus, we find

$$\dot{T}_{ab}{}^c = -2\,\dot{\nabla}_{[ab]}{}^c = 0. \tag{1.10}$$

These two constraints actually determine $\dot{\nabla}_{ab}{}^c$ uniquely. To see this, we lower the index on the torsion constraint using the background metric and write

$$\dot{\nabla}_{abc} = -\nabla_a\,\dot{g}_{bc} - \dot{\nabla}_{acb} = -\nabla_a\,\dot{g}_{bc} - \dot{\nabla}_{cab}, \tag{1.11}$$

where we have used the metricity constraint in the first equality and the torsion constraint in the second. The resulting connection perturbation on the right has its indices cyclically permuted once compared to that on the left. Therefore, applying this result twice more gives

$$\begin{aligned}\dot{\nabla}_{abc} &= -\nabla_a\,\dot{g}_{bc} - \big(-\nabla_c\,\dot{g}_{ab} - \dot{\nabla}_{bca}\big)\\ &= -\nabla_a\,\dot{g}_{bc} + \nabla_c\,\dot{g}_{ab} - \nabla_b\,\dot{g}_{ac} - \dot{\nabla}_{abc}\end{aligned} \tag{1.12}$$

The connection perturbation on the right now has its indices back in the original perturbation, but occurs with a minus sign. Collecting these perturbations on the left and dividing through by the factor of two gives the result

$$\dot{\nabla}_{ab}{}^c = -\tfrac{1}{2}\,g^{cm}\,\big(2\,\nabla_{(a}\,\dot{g}_{b)m} - \nabla_m\,\dot{g}_{ab}\big) = -\tfrac{1}{2}\,g^{cm}\,\big(2\,\nabla_{[b}\,\dot{g}_{m]a} + \nabla_a\,\dot{g}_{bm}\big), \tag{1.13}$$

where we have once again raised the last index with the background metric to put the connection perturbation in its natural tensorial form. Note that the derivative operators on the right here are those compatible with the background metric.

Calculating the perturbation in the Riemann tensor is slightly easier. Since the metric connections are all torsion-free, we have

$$R_{abc}{}^d\,\omega_d := 2\,\nabla_{[a}\,\nabla_{b]}\,\omega_c \tag{1.14}$$

for each λ and for an arbitrary co-vector field ω_a, which we may assume does not vary with λ. Differentiating both sides of this definition with respect to λ, we see that the derivative on the right may act on either of two copies of the connection. Acting on the first will produce two connection perturbation tensors, since this derivative acts on a two-index tensor, while acting on the second produces only one:

$$\begin{aligned}\dot{R}_{abc}{}^{d}\,\omega_d &= 2\,\dot{\nabla}_{[ab]}{}^{d}\,\nabla_d\,\omega_c + 2\,\dot{\nabla}_{[a|c|}{}^{d}\,\nabla_{b]}\,\omega_d + 2\,\nabla_{[a}\left(\dot{\nabla}_{b]c}{}^{d}\,\omega_d\right)\\ &= 2\,\dot{\nabla}_{[a|c|}{}^{d}\,\nabla_{b]}\,\omega_d + 2\,\nabla_{[a}\,\dot{\nabla}_{b]c}{}^{d}\,\cdot\omega_d + 2\,\dot{\nabla}_{[b|c|}{}^{d}\,\nabla_{a]}\,\omega_d.\end{aligned} \tag{1.15}$$

In the second equality here, we have dropped the first term because ∇_a remains torsion-free, and expanded the third term using the Leibniz rule. The first and third terms in the result cancel leaving only the second, which is algebraic at each point in the arbitrary co-vector field ω_d. We therefore find the local equation

$$\dot{R}_{abc}{}^{d} = 2\,\nabla_{[a}\,\dot{\nabla}_{b]c}{}^{d} \tag{1.16}$$

of tensor fields.

We now want to deduce the perturbation in the Einstein tensor. First, note that its definition can be written in the form

$$G_{ab} = R_{ab} - \tfrac{1}{2} R\, g_{ab} = \left(\delta^i_a\,\delta^j_b - \tfrac{1}{2}\, g^{ij}\, g_{ab}\right) R_{ij}. \tag{1.17}$$

We refer to the tensor in braces here as the ***trace-reversal operator*** since, in four spacetime dimensions, that is exactly what it does to a two-index tensor. When we differentiate this result with respect to λ, the derivatives can either act on this operator or on the Ricci tensor. In the former case, the identity operators δ^m_n do not depend on λ, but the metric g_{ab} and its inverse g^{ij} do, and there is a notational subtlety when it acts on the the inverse that we must discuss.

For each value of λ, the definition of the inverse metric g^{ab} is that

$$g^{ab}\, g_{bc} = \delta^a_c. \tag{1.18}$$

Differentiating with respect to λ yields

$$g_{bc}\,\frac{\mathrm{d}g^{ab}}{\mathrm{d}\lambda} + g^{ab}\,\frac{\mathrm{d}g_{bc}}{\mathrm{d}\lambda} = 0 \tag{1.19}$$

because the identity operator is λ-independent. We can solve this equation for the derivative of the inverse metric by moving the other term to the right and multiplying through by the inverse of the remaining (background) metric on the left:

$$\frac{\mathrm{d}g^{ab}}{\mathrm{d}\lambda} = -g^{am}\,\frac{\mathrm{d}g_{mn}}{\mathrm{d}\lambda}\,g^{nb} =: -g^{am}\,\dot{g}_{mn}\,g^{nb}. \tag{1.20}$$

The minus sign on the right here shows that we must exercise some care regarding the symbol $\dot{g}^{ab}$. In particular, it could denote *either* the perturbation of the inverse metric *or* the ordinary, covariant metric perturbation $\dot{g}_{ab}$ with its indices raised using the background metric g^{ab}. It *cannot* denote both simply because of the minus sign we have found here and, in fact, we prefer the latter notation. Thus, we write

$$\frac{\mathrm{d}g^{ab}}{\mathrm{d}\lambda} := -\dot{g}^{ab}. \tag{1.21}$$

More generally, we denote with dots the perturbations of tensors with their *natural* index structures that arise in our theories. Those indices may subsequently be raised or lowered using the background metric, but the definition of the dot operation refers to the natural index placement. Thus, for example, $\dot{\nabla}_{abc}$ above refers to the metric perturbation $\dot{\nabla}_{ab}{}^{c}$ with its last index lowered using the background metric and *not* to the λ-derivative of $\Gamma_{ab}{}^{m}(\lambda)\, g_{mc}(\lambda)$, where $\Gamma_{ab}{}^{c}(\lambda)$ describes the difference between $\nabla_a(\lambda)$ and some fixed coordinate connection ∂_a. The two are generally different.

Returning to our discussion of the Einstein tensor, we now can write

$$\dot{G}_{ab} = -\tfrac{1}{2}\big(-\dot{g}^{ij}\, g_{ab} + g^{ij}\,\dot{g}_{ab}\big)\, R_{ij} + \big(\delta^i_a\,\delta^j_b - \tfrac{1}{2}\, g^{ij}\, g_{ab}\big)\,\dot{R}_{ij}. \tag{1.22}$$

Note the sign in the first term. To proceed, We must calculate the perturbation of the Ricci tensor. Note that the identity operator used δ^b_d used to contract the Riemann perturbation above passes through both the background covariant derivative ∇_a and the λ-derivative. Thus, we can write

$$\dot{R}_{ab} = 2\,\nabla_{[a}\,\dot{\nabla}_{m]b}{}^{m} = -g^{mn}\,\nabla_{[a}\,\big(2\,\nabla_{[b}\,\dot{g}_{n]m]} + \nabla_{m]}\,\dot{g}_{bn}\big). \tag{1.23}$$

We have inserted our expression for the connection perturbation in the second equality, with the convention that the indices within the nested anti-symmetrization do no participate in the outer one. Expanding this gives four second-derivative terms from the first term, and two Riemann-curvature terms from the second:

$$\begin{aligned}\dot{R}_{ab} = -\tfrac{1}{2}\, g^{mn}\,\big(&\nabla_a\,\nabla_b\,\dot{g}_{nm} - \nabla_a\,\nabla_n\,\dot{g}_{bm} - \nabla_m\,\nabla_b\,\dot{g}_{na} + \nabla_m\,\nabla_n\,\dot{g}_{ba}\\ &+ R_{amb}{}^{d}\,\dot{g}_{dn} + R_{amn}{}^{d}\,\dot{g}_{bd}\big)\end{aligned} \tag{1.24}$$

To show explicitly that the Ricci perturbation is symmetric in its two indices, which it certainly must be, it will be convenient to commute the two derivative operators in the third term here so that it more closely resembles the second. This produces two more curvature terms:

$$\begin{aligned}\dot{R}_{ab} = -\tfrac{1}{2}\, g^{mn}\,\big(&\nabla_m\,\nabla_n\,\dot{g}_{ab} - \nabla_a\,\nabla_n\,\dot{g}_{mb} - \nabla_b\,\nabla_m\,\dot{g}_{na} + \nabla_a\,\nabla_b\,\dot{g}_{mn}\\ &- R_{mbn}{}^{d}\,\dot{g}_{da} - R_{mba}{}^{d}\,\dot{g}_{nd} + R_{amb}{}^{d}\,\dot{g}_{dn} + R_{amn}{}^{d}\,\dot{g}_{bd}\big).\end{aligned} \tag{1.25}$$

We now act with the inverse metric to get our final result:

$$\dot{R}_{ab} = -\tfrac{1}{2}\,\nabla_m\,\nabla^m\,\dot{g}_{ab} + \nabla_{(a}\,\nabla^m\,\big(\dot{g}_{b)m} - \tfrac{1}{2}\dot{g}\, g_{b)m}\big) + R_{(a}{}^{m}\,\dot{g}_{b)m} - R_a{}^{m}{}_b{}^{n}\,\dot{g}_{mn}. \tag{1.26}$$

This is the first-order perturbation in the Ricci tensor induced by a given first-order metric perturbation.

When we insert the Ricii perturbation into the expression above for the Einstein perturbation, note that the trace-reversal operator will pass through the derivatives in the first term here, and therefore that both sets of derivative terms can be written simply in terms of the ***trace-reversed metric perturbation***[1]

$$h_{ab} := \dot{g}_{ab} - \tfrac{1}{2}\dot{g}\, g_{ab} \quad \text{with} \quad \dot{g} := \dot{g}_m{}^m. \tag{1.27}$$

In addition, note that we can replace the metric perturbations $\dot{g}_{mn}$ with their trace-reversals h_{mn} in the curvature terms since the extra terms are both proportional to $\dot{g}\, R_{ab}$, and cancel. Likewise, we can replace the metric perturbations in the derivative of the trace-reversal operator with their trace-reversals because the extra terms cancel. This gives

$$\begin{aligned}\dot{G}_{ab} = \tfrac{1}{2}\big(h^{ij}\, g_{ab} - g^{ij}\, h_{ab}\big)\, R_{ij} - \tfrac{1}{2}\,\nabla_m\,\nabla^m\, h_{ab} \\ + \big(\delta^i_a\,\delta^j_b - \tfrac{1}{2}\, g^{ij}\, g_{ab}\big)\,\big(\nabla_{(i}\,\nabla^m\, h_{j)m} + R_{(i}{}^m\, h_{j)m} - R_i{}^m{}_j{}^n\, h_{mn}\big).\end{aligned} \tag{1.28}$$

Finally, we can simplify this slightly by bringing the middle term in the final braces out to cancel the first term, leaving

$$\begin{aligned}\dot{G}_{ab} = -\tfrac{1}{2}\,\nabla_m\,\nabla^m\, h_{ab} + G_{(a}{}^m\, h_{b)m} \\ + \big(\delta^i_a\,\delta^j_b - \tfrac{1}{2}\, g^{ij}\, g_{ab}\big)\big(\nabla_{(i}\,\nabla^m\, h_{j)m} - R_i{}^m{}_j{}^n\, h_{mn}\big).\end{aligned} \tag{1.29}$$

Note that the second term in this final result involved the Einstein tensor of the background spacetime, which is determined by the background source distribution via the Einstein equations. However, the last term involves the *full* Riemann tensor, which is not *entirely* determined by the background source.

1.2 FIRST-ORDER POST-MINKOWSKI GRAVITY

An important application of the linearized formalism above arises when *all* gravitational sources in a region of spacetime are weak. In this case, we can set

$$\mathring{g}_{ab} = \eta_{ab} \quad \text{and} \quad \mathring{\nabla}_a = \partial_a. \tag{1.30}$$

That is, we take the background metric to be the Minkowski metric, for which the compatible connection is the flat inertial coordinate derivative. This causes the curvature terms to drop out of the linearized field equation, and we can write

$$\dot{G}_{ab} = -\tfrac{1}{2}\,\partial_m\,\partial^m\, h_{ab} + \partial_{(a}\,\partial^m\, h_{b)m} - \tfrac{1}{2}\,\eta_{ab}\,\partial^m\,\partial^n\, h_{mn} = 8\pi\,\dot{T}_{ab}. \tag{1.31}$$

[1] Our notation here is slightly non-standard. Our original metric perturbation $\dot{g}_{ab}$ is often denoted h_{ab}, a notation we use here to denote the trace-reversed perturbation. This trace-reversed object is conventionally denoted $\bar{h}_{ab}$. That is, our $\dot{g}_{ab}$ is their h_{ab}, and our h_{ab} is their $\bar{h}_{ab}$. Hopefully, this will not cause too much confusion.

This is the first-order ***post-Minkowski field equation***.

There are many similarities between the post-Minkowski theory and Maxwell theory on flat spacetime. For instance, if we take the divergence of both sides of the post-Minkowski equation, we find that

$$\partial^a \dot{G}_{ab} = -\tfrac{1}{2}\,\partial_m\,\partial^m\,\partial^a\,h_{ab} + \partial^a\,\partial_{(a}\,\partial^m\,h_{b)m} - \tfrac{1}{2}\,\partial_b\,\partial^m\,\partial^n\,h_{mn} = 0 \tag{1.32}$$

on the left. Expanding the symmetrization in the middle term produces terms that cancel the first and last exactly. Note that this is an identity; it does not rely on h_{ab} satisfying the post-Minkowski field equation. This is not a coincidence. Indeed, the contracted Bianchi identity demands that

$$\nabla^a\,G_{ab} = g^{mn}\,\nabla_m\,G_{nb} = 0 \tag{1.33}$$

for all λ. Taking a derivative of this with respect to λ implies that

$$g^{mn}\,\nabla_m\,\dot{G}_{nb} + 2\,g^{mn}\,\dot{\nabla}_{m(n}{}^{c}\,G_{b)c} - \dot{g}^{mn}\,\nabla_m\,G_{nb} = 0. \tag{1.34}$$

When the background spacetime satisfies the vacuum Einstein equations $G_{mn} = 0$, as Minkowski spacetime certainly does, the last two terms vanish. Therefore, the result above is really a manifestation of the contracted Bianchi identity in our post-Minkowski theory. Like that identity in the full theory, it implies that post-Minkowski gravity is over-determined. The source must satisfy

$$\partial^a\,\dot{T}_{ab} = 0 \tag{1.35}$$

for any solution of the field equation to exist. That is, for the post-Minkowski equation to be soluble, the energy and momentum of its source must be conserved. This result seems a little troubling at first, since it evidently implies that, for example, particle sources of gravitational perturbations may only move *inertially* through Minkowski spacetime. That is, they must move along straight lines. On the other hand, many situations of interest would involve, for example, orbiting stars, which certainly do not move along inertial trajectories. Happily, there is no real problem, however. It is necessary only that the violation of this integrability condition for the post-Minkowski theory be *higher-order* in perturbation theory. That is, for example, particle sources may accelerate, but that acceleration must be at least quadratic in the perturbation parameter λ. Thus, we need not solve the integrability condition exactly, only to the order of perturbation at which we work.

The post-Minkowski field equations are also *under*-determined, again like the Maxwell equations. In this case, however, the degeneracy is not so obvious. Luckily, it does emerge readily from the general diffeomorphism freedom of general relativity. Consider two families of metrics

$$g_{ab}(\lambda) \quad \text{and} \quad \tilde{g}_{ab}(\lambda) := \Phi(\lambda)\cdot g_{ab}(\lambda), \tag{1.36}$$

where $\Phi(\lambda)$ is a smooth, one-parameter family of diffeomorphisms of the manifold M with $\Phi(0)$ the identity diffeomorphism. For each λ, as we have discussed, these two metrics are physically equivalent; they differ only by how we label the points of spacetime. When we differentiate the second family of metrics with respect to λ, the derivative can either act on the metric *or the diffeomorphism.* In the latter case, the effect is *exactly* a Lie derivative along the vector field ϕ^a generating the diffeomorphisms $\Phi(\lambda)$:

$$\dot{\tilde{g}}_{ab} = \frac{\mathrm{d}}{\mathrm{d}\lambda}\big[\Phi(\lambda)\cdot g_{ab}(0)\big] + \Phi(0)\cdot \dot{g}_{ab} = \mathscr{L}_\phi\, \eta_{ab} + \dot{g}_{ab} = \dot{g}_{ab} + 2\,\partial_{(a}\,\phi_{b)}. \tag{1.37}$$

Thus, we can read off a guess at the action of a gauge transformation in post-Minkowski gravity on the metric perturbation $\dot{g}_{ab}$. We deduce thereby the corresponding action on the trace-reversed perturbation:

$$h_{ab} \mapsto \tilde{h}_{ab} := h_{ab} + 2\,\partial_{(a}\,\phi_{b)} - \eta_{ab}\,\partial_c\,\phi^c. \tag{1.38}$$

To check that this is indeed a gauge transformation, we insert the difference of these tensors into the left side of the post-Minkowski field equation to find

$$\begin{aligned}
-\tfrac{1}{2}\,\partial_m\,\partial^m\,\big(2\,\partial_{(a}\,\phi_{b)} - \eta_{ab}\,\partial_c\,\phi^c\big)& \\
+\,\partial_{(a}\,\partial^m\,\partial_{b)}\,\phi_m + \partial_{(a}\,\partial^m\,\partial_{|m|}\,\phi_{b)} - \partial_{(a}\,\partial^m\,\big(\eta_{b)m}\,\partial_c\,\phi^c\big)& \\
-\tfrac{1}{2}\,\eta_{ab}\,\partial^m\,\partial^n\,\big(2\,\partial_{(m}\,\phi_{n)} - \eta_{mn}\,\partial_c\,\phi^c\big) &= 0.
\end{aligned} \tag{1.39}$$

The two terms on the last line here differ by a sign and a factor of two, and together cancel the last term on the first line. The first term on the first line likewise cancels the middle term on the second, and the first and last terms on the second line cancel one another. Thus, post-Minkowski gravity has a gauge ambiguity similar to that of Maxwell theory, though the gauge transformations are generated by vector fields ϕ^a rather than scalar functions ψ.