Lecture 4

The Principle of Equivalence

What is gravity?

Uniformly Accelerated Motion

world-line $w(\lambda)$

$\rightsquigarrow$ 4-velocity $u(\lambda) = \frac{dw}{d\tau}(\lambda)$

$\rightsquigarrow$ 4-acceleration $a(\lambda) = \frac{du}{d\tau}(\lambda)$

$\rightsquigarrow$ 4-jerk $j(\lambda) = \frac{da}{d\tau}(\lambda)$

Newtonian physics: $j = 0 \leftarrow$ Does not work in SR!

$$u(\lambda) \cdot u(\lambda) = -c^2$$

$$\Rightarrow \quad u(\lambda) \cdot a(\lambda) = 0$$

$$\Rightarrow \quad a(\lambda) \cdot a(\lambda) + u(\lambda) \cdot j(\lambda) = 0$$

$a(\lambda)$ is space-like, so unless $a(\lambda) = 0$, $\|a(\lambda)\|^2 > 0$

$\Rightarrow j(\lambda) \neq 0 \leftarrow$ must have a component along $u(\lambda)$

Uniform acceleration in SR makes the minimal assumption that $j(\lambda) \propto u(\lambda)$

$$j(\lambda) = \alpha\, u(\lambda)$$

$$\leadsto \|a(\lambda)\|^2 + \alpha \overbrace{\|u(\lambda)\|^2}^{-c^2} = 0$$

$$\Rightarrow \alpha = \frac{\|a(\lambda)\|^2}{c^2}$$

So $\displaystyle j(\lambda) = \frac{\|a(\lambda)\|^2}{c^2}\, u(\lambda)$

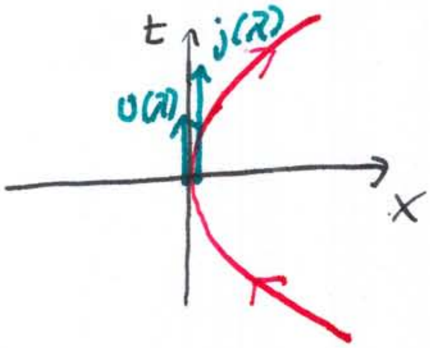
$$\Rightarrow \underbrace{a(\lambda)\cdot j(\lambda)}_{=\ \frac{1}{2}\frac{d}{d\tau}\|a(\lambda)\|^2} = 0$$

$$\Rightarrow \|a(\lambda)\|^2 = g^2 = \text{const.}$$

positive b/c $a(\lambda)$ space-like

$$\Rightarrow j(\lambda) = \boxed{\frac{d^2}{d\tau^2}\, u(\lambda) = \frac{g^2}{c^2}\, u(\lambda)}$$

Note: In uniform acceleration, the spatial projection of $j(\lambda)$ vanishes in the instantaneously co-moving frame: $\vec{j}(\lambda) = 0$



$$\vec{j}(\lambda) = \frac{d^3}{d\tau^3}\vec{x}(\lambda)$$

$$= \left(\frac{dt}{d\tau}\frac{d}{dt}\right)^3 \vec{x}(\lambda)$$

$$\gamma(\lambda) = \frac{1}{\sqrt{1 - v^2(\lambda)/c^2}}$$

$$\left(\gamma \frac{d}{dt}\right)^3 = \gamma^3 \frac{d^3}{dt^3} + 3\gamma^2\dot{\gamma}\frac{d^2}{dt^2} + \gamma(\dot{\gamma}^2 + \gamma\ddot{\gamma})\frac{d}{dt}$$

$=0$ in icio

$$\vec{j}(\lambda) = \gamma^3(\lambda)\frac{d^3\vec{x}}{dt^3}(\lambda) + 3\gamma^2(\lambda)\dot{\gamma}(\lambda)\frac{d^2\vec{x}}{dt^2}(\lambda) + \gamma(\lambda)\left(\dot{\gamma}^2(\lambda) + \gamma(\lambda)\ddot{\gamma}(\lambda)\right)\frac{d\vec{x}}{dt}(\lambda)$$

$=0$ in icio

$\therefore$ uniform acceleration

$\Updownarrow$

icio sees no (Newtonian) jerk.

Need to solve ODE:

$$\frac{d^2}{d\tau^2} u(\lambda) = \frac{g^2}{c^2} u(\lambda)$$

$$\rightsquigarrow \quad u(\lambda) = u_0 \cosh\left(\frac{g}{c}(\tau - \tau_0)\right) + \frac{c}{g} a_0 \sinh\left(\frac{g}{c}(\tau - \tau_0)\right)$$

u_0 = 4-velocity at $\tau = \tau_0$

a_0 = 4-acceleration at $\tau = \tau_0$

$$\begin{aligned}
\|u(\lambda)\|^2 &= \|u_0\|^2 \cosh^2\left(\frac{g}{c}(\tau-\tau_0)\right) \\
&\quad + \|a_0\|^2 \frac{c^2}{g^2} \sinh^2\left(\frac{g}{c}(\tau-\tau_0)\right) \\
&\quad + 2\, u_0 \cdot a_0 \frac{c}{g} \cosh\left(\frac{g}{c}(\tau-\tau_0)\right) \sinh(\cdots) \\
&= -c^2 \cosh^2(\cdots) + c^2 \sinh^2(\cdots) \\
&= -c^2
\end{aligned}$$

$\Rightarrow$ $u(\tau)$ is proper-time parameterized

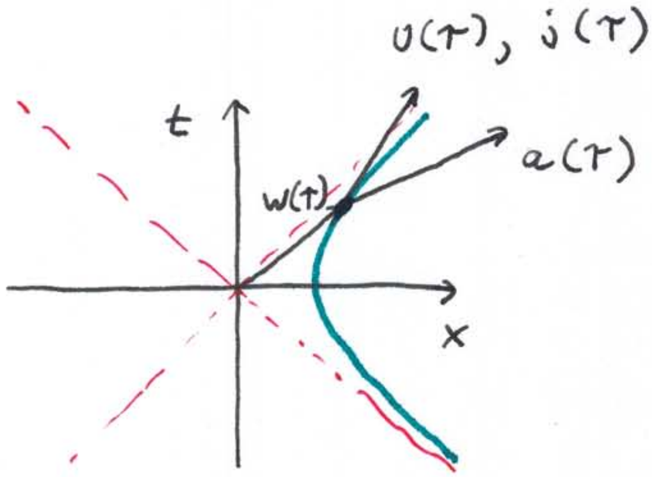
Integrate one more time

$$w(\tau) = w_0 + \frac{c}{g} u_0 \sinh\left(\tfrac{g}{c}(\tau - \tau_0)\right) + \frac{c^2}{g^2} a_0 \cosh\left(\tfrac{g}{c}(\tau - \tau_0)\right)$$

Take $w_0 = 0$, $\tau_0 = 0$

$$\|w(\tau)\|^2 = \frac{c^2}{g^2} \cdot -c^2 \sinh^2(\cdots) + \frac{c^4}{g^4} \cdot g^2 \cosh^2(\cdots) = \frac{c^4}{g^2}$$

constant

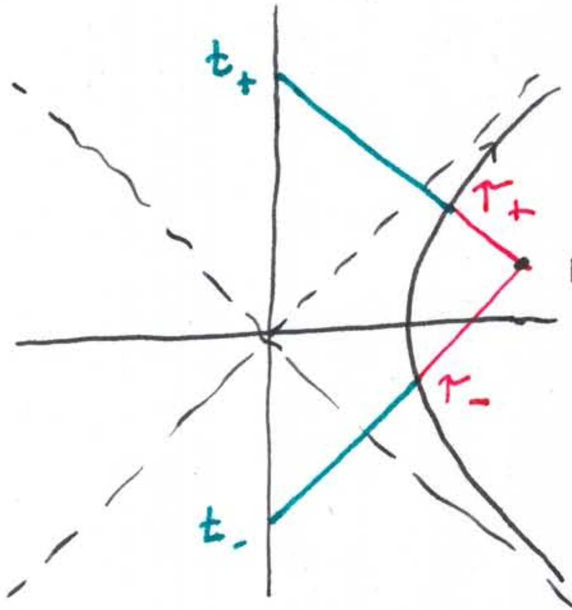


Uniform acceleration is hyperbolic motion in the u-a plane (2-d.)

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \frac{c}{g} \sinh\left(\frac{g}{c}(\tau - \tau_0)\right) \\ \frac{c^2}{g} \cosh\left(\frac{g}{c}(\tau - \tau_0)\right) \end{pmatrix}$$

$$x(t) = \frac{c^2}{g}\sqrt{1 + \left(\frac{g}{c} t\right)^2}$$

Radio-Coordinates of an Accelerating Observer



$E = (t, x) = (\tau, \xi)$

$$t = \frac{t_+ + t_-}{2}$$

$$x = \frac{t_+ - t_-}{2}\, c$$

$$\tau = \frac{\tau_+ + \tau_-}{2}$$

$$\xi = \frac{\tau_+ - \tau_-}{2}\, c$$

$$t_+ = t(\tau_+) + \frac{x(\tau_+)}{c}$$

$$= \frac{c}{g}\left[\sinh \frac{g\tau_+}{c} + \cosh \frac{g\tau_+}{c}\right]$$

$$= \frac{c}{g}\, e^{g\tau_+/c}$$

$$t_- = t(\tau_-) - \frac{x(\tau_-)}{c} = -\frac{c}{g}\, e^{-g\tau_-/c}$$

$$t = \frac{c}{2g}\left(e^{g\tau_+/c} - e^{-g\tau_-/c}\right) = \frac{c}{g}\, e^{\frac{g\xi}{c^2}} \sinh \frac{g\tau}{c}$$

$$x = \frac{c^2}{2g}\left(e^{g\tau_+/c} + e^{-g\tau_-/c}\right) = \frac{c^2}{g}\, e^{\frac{g\xi}{c^2}} \cosh \frac{g\tau}{c}$$

$$\tau_\pm = \tau \pm \frac{\xi}{c}$$

Inverse transformation:

$$t_\pm = \pm \frac{c}{g} e^{\pm g \tau_\pm / c}$$

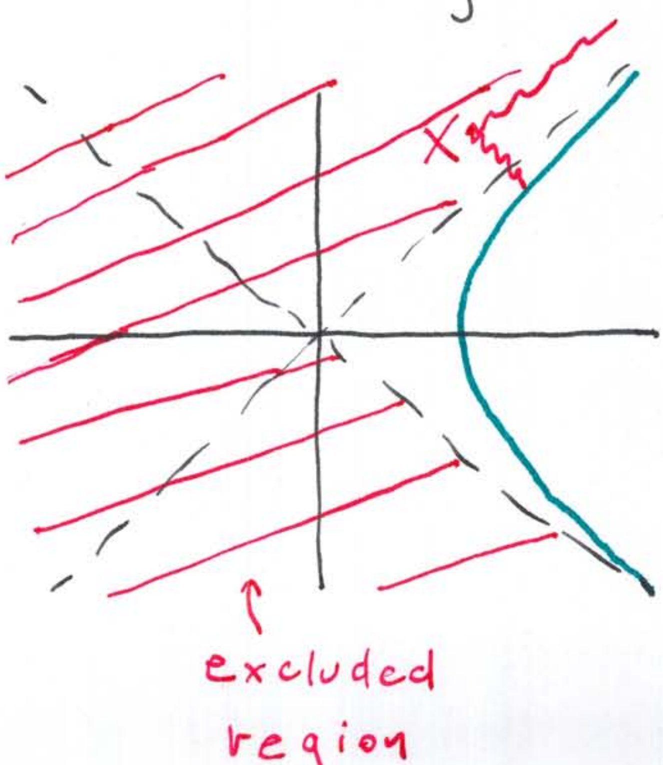
$$\Rightarrow \quad \tau_\pm = \pm \frac{c}{g} \ln\left(\pm \frac{g}{c} t_\pm\right)$$

$$\Rightarrow \quad \tau \pm \frac{\xi}{c} = \pm \frac{c}{g} \ln\left(\pm \frac{g}{c}\left(t \pm \frac{x}{c}\right)\right)$$

$$= \pm \frac{c}{g} \ln\left(\frac{g}{c^2}(x \pm ct)\right)$$

$$\rightsquigarrow \quad \tau = \frac{c}{2g} \ln \frac{x+ct}{x-ct}$$

$$\xi = \frac{c^2}{2g} \ln\left(\frac{g^2}{c^4}(x^2 - c^2 t^2)\right)$$



Radio coordinates cover <u>only</u> the <u>Rindler wedge</u> in Minkowski spacetime.

The metric in accelerating radio coordinates:

$$t = \frac{c}{g} e^{g\xi/c^2} \sinh\left(\frac{g\tau}{c}\right)$$

$$x = \frac{c^2}{g} e^{g\xi/c^2} \cosh\left(\frac{g\tau}{c}\right)$$

$$-c^2 dt^2 + dx^2 = e^{2g\xi/c^2}\left(-d\tau^2 + d\xi^2\right)$$

↑

exercise!