Maximal Symmetry

Killing Fields: ξ^a

① $\mathcal{L}_\xi g_{ab} = 2\nabla_{(a}\xi_{b)} = 0$

$$\Rightarrow \quad \nabla_a \xi_b = \nabla_{[a}\xi_{b]}$$

② $\nabla_a \nabla_b \xi_c = -\nabla_a \nabla_c \xi_b$

$$= -R_{acb}{}^{d}\xi_d - \nabla_c \nabla_a \xi_b$$

$$= R_{cab}{}^{d}\xi_d - \nabla_c \nabla_a \xi_b$$

$$= R_{cab}{}^{d}\xi_d - R_{b\underset{ca}{\cancel{ac}}}{}^{d}\xi_d + \nabla_b \nabla_c \xi_a$$

$$= \left(R_{cab}{}^{d} - R_{b\underset{ca}{\cancel{ac}}}{}^{d} + R_{abc}{}^{d}\right)\xi_d - \nabla_a \nabla_b \xi_c$$

$$2\nabla_a \nabla_b \xi_c = -2R_{b\underset{ca}{\cancel{ac}}}{}^{d}\xi_d$$

$$\nabla_a \nabla_b \xi_c = \xi^d R_{dabc}$$

So, Killing fields satisfy

$$\nabla_a \xi_b = L_{ab} = L_{[ab]}$$

$$\nabla_a L_{bc} = \xi^d R_{dabc}$$

$\Rightarrow$ If two Killing vectors ξ^a and $\tilde{\xi}^a$ have

$$\left.\begin{array}{l} \xi^a = \tilde{\xi}^a \\ L_{ab} = \tilde{L}_{ab} \end{array}\right\}$$ at <u>any</u> one point of space

$\rightsquigarrow$ Call (ξ_a, L_{bc})

<u>Killing data</u> at a point.

Analogy: V^a, $\tilde{V}^a$

$$\nabla_a V^b = 0 = \nabla_a \tilde{V}^b$$

<u>and</u> $V^a = \tilde{V}^a$ at p

$\Rightarrow$ $V^a = \tilde{V}^a$ <u>everywhere</u>

<u>Killing Transport</u>

$$K_v(\xi_a, L_{bc}) = 0$$

$$= \Big(v^m(\nabla_m \xi_a - L_{ma}),$$

$$v^m(\nabla_m L_{bc} - \xi^d R_{dmbc}) \Big)$$

obstruction to integrability of Killing transport.

$$\rightsquigarrow \boxed{[K_v, K_w] - K_{[v,w]}}$$

$$K_v(\xi_a, L_{bc}) := \Big(\nabla_v \xi_a - v^m L_{ma}, \\ \nabla_v L_{bc} - \xi^d v^m R_{dmbc} \Big)$$

$$K_w K_v (\xi_a, L_{bc}) =$$

$$= \Big(\nabla_w (\nabla_v \xi_a - v^m L_{ma}) \\ - w^n (\nabla_v L_{na} - \xi^d v^m R_{dmna}), \\ \nabla_w (\nabla_v L_{bc} - \xi^d v^m R_{dmbc}) \\ \cancel{+ \xi^d w^n} \\ - (\nabla_v \xi^d - v^m L_m{}^d) w^n R_{dnbc} \Big)$$

$$K_w K_v (\xi_a, L_{bc}) =$$

$$= \Big(\nabla_w \nabla_v \xi_a - \nabla_w (V^m L_{ma}) - W^n \nabla_v L_{na}$$

$$+ \xi^d V^m W^n R_{dmna},$$

$$\nabla_w \nabla_v L_{bc} - \nabla_w (\xi^d V^m R_{dmbc})$$

$$- \nabla_v \xi^d \cdot W^n R_{dnbc} + V^m L_m{}^d W^n R_{dnbc} \Big)$$

$$K_{[v,w]} (\xi_a, L_{bc})$$

$$=$$

$$\Big([K_v, K_w] - K_{[v,w]} \Big) (\xi_a, L_{bc})$$

$$= \Big(0,\ V^m W^n \Big(\xi^d \nabla_d R_{mnbc}$$

$$V^m W^n \mathcal{L}_\xi R_{mnbc} \rightarrow - \cancel{4} 2 L_{[m}{}^d R_{n]dbc}$$

$$- 2 L_{[b}{}^d R_{|mn|c]d} \Big) \Big)$$

Maximal Symmetry

The dimension of the space of Killing data at a point is

$$\left(\xi_a,\ L_{bc}\right)$$

$$d + \frac{d(d-1)}{2} = \frac{d(d+1)}{2}$$

maximal symmetry in dimension d.

$$\mathcal{L}_{\alpha\xi + \alpha'\xi'}\, g_{ab} = 2\nabla_{(a}\left(\alpha\xi_{b)} + \alpha'\xi'_{b)}\right)$$

Euclidean space is a space of maximal symmetry.

$$J_{ab} = -\int_{\Sigma} 2\, x_{[a} T_{b]c}\, \hat{n}^c \, d\Sigma$$

$$\hat{n}^a J_{ab} = \int_{\Sigma} x_b T_{an} - x_n P_b \, d\Sigma$$