Lecture 21

Physics of Black Holes

Minkowski Compactified

The physical Minkowski metric is

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

$$= -du\,dv + \frac{1}{4}(v-u)^2 d\Omega^2$$

$u := t - r \qquad t = \frac{1}{2}(v+u)$

$v := t + r \qquad r = \frac{1}{2}(v-u)$

We choose the conformal factor

$$\Omega^2 = \frac{1}{4}(1+u^2)(1+v^2)$$

This gives the conformal metric

$$d\mathring{s}^2 = \frac{-4\,du\,dv}{(1+u^2)(1+v^2)} + \frac{(v-u)^2 d\Omega^2}{(1+u^2)(1+v^2)}$$

Note that the components here are well-behaved as $u, v \to \infty$.

We now bring infinity to a finite coordinate distance by setting

$$u = \tan U \qquad du = (1+u^2)\, dU$$

$$v = \tan V \qquad dv = (1+v^2)\, dV$$

Then we find

$$d\mathring{s}^2 = -4\, dU\, dV + \frac{(\tan V - \tan U)^2}{\sec^2 U \sec^2 V}\, d\Omega^2$$

$$= -4\, dU\, dV + \sin^2(V-U)\, d\Omega^2$$

Finally, we introduce coordinates

$$\tau := U + V \qquad \psi := V - U$$

$$\rightsquigarrow d\mathring{s}^2 = -d\tau^2 + d\psi^2 + \sin^2\psi\, d\Omega^2$$

This metric describes the round unit 3-sphere S^3 crossed with time τ. It is a cylinder.

Minkowski spacetime is conformally embedded into a subset of this cylinder. What are its boundaries?

$$-\infty < \tan U := u := t - r < \infty$$

$$-\infty < \tan V := v := t + r < \infty$$

But the radius is positive, so

$$r \geq 0 \quad \Rightarrow \quad v \geq u \quad \Rightarrow \quad V \geq U$$

because the tangent is monotonic. We therefore have

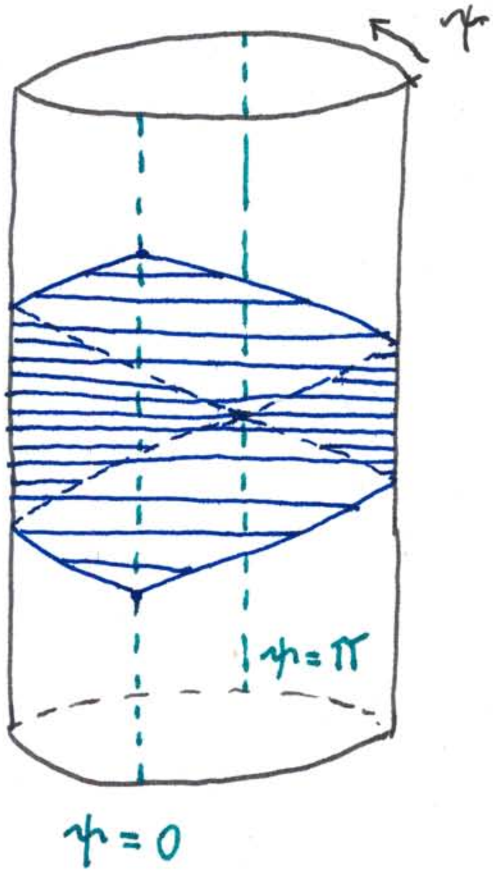
$$-\frac{\pi}{2} < \frac{\tau - \psi}{2} := U \leq V =: \frac{\tau + \psi}{2} < \frac{\pi}{2}$$

The inner inequality clearly demands $\psi \geq 0$, and we can rewrite the outer ones as

$$\begin{matrix} \psi + \tau < \pi \\ \psi - \tau < \pi \end{matrix} \quad \Rightarrow \quad \psi < \pi - |\tau|$$

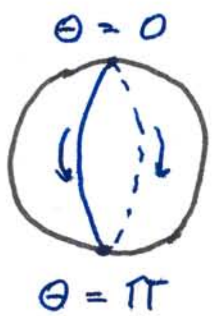
$$
\begin{tikzcd}
 & -\pi < \tau - \psi \le \tau + \psi < \pi \arrow[dl] \arrow[dr] & \\
\psi - \tau < \pi & \psi \ge 0 \arrow[u] & \psi + \tau < \pi \\
\psi < \pi + \tau \arrow[dr] & & \psi < \pi - \tau \arrow[dl] \\
 & 0 \le \psi < \pi - |\tau| &
\end{tikzcd}
$$

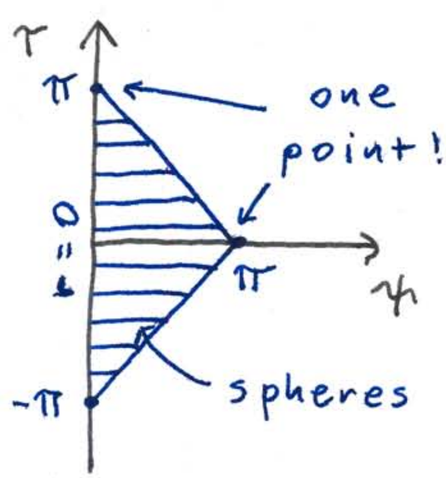
We have therefore conformally embedded Minkowski spacetime into the region



$$-\pi < \tau < \pi$$

$$0 \leq \psi < \pi - |\tau|$$





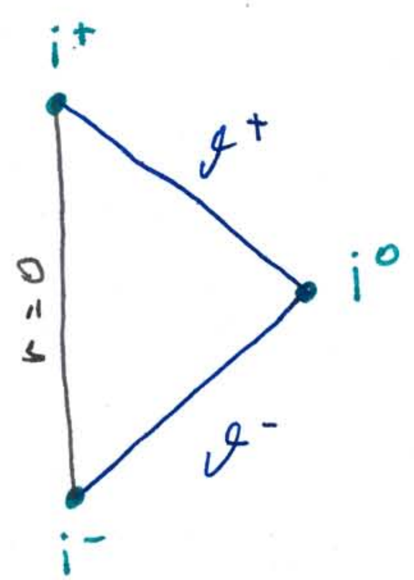
<u>Penrose</u>

<u>diagram</u>

The Boundary of Spacetime

We compactify Minkowski spacetime by adding the points on the boundary in the Penrose (conformal) diagram.

These boundary points divide naturally into physically distinct sets:



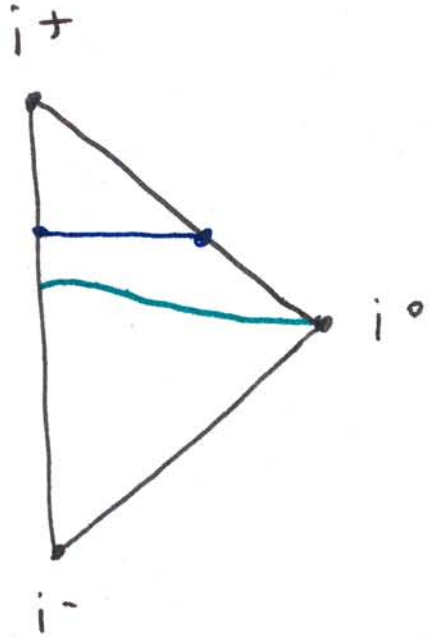
cylinder $S^2 \times \mathbb{R}$

$\mathscr{I}^\pm$ = future/past null infinity

$i^\pm$ = future/past time-like infinity

i^0 = spatial infinity

single points



i^+
i^0
i^-

These added boundary points have the following physical interpretations:

i^+: "endpoint" of future-directed time-like geodesics of infinite length in the <u>physical</u> metric.

i^-: "endpoint" of past-directed time-like geodesics

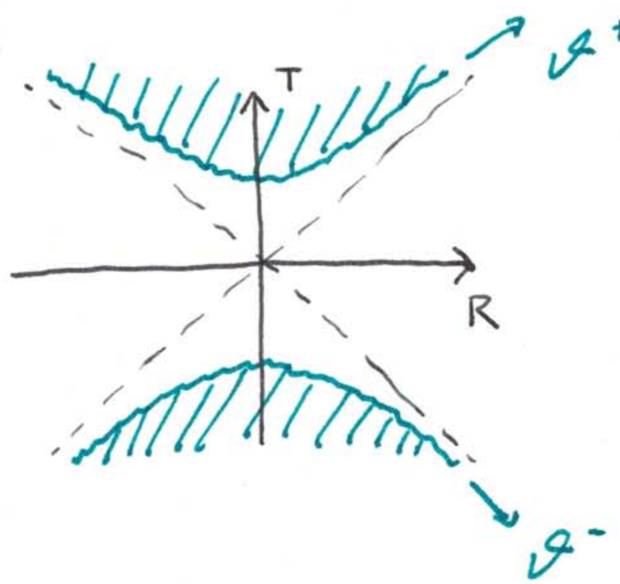
$\mathscr{I}^+$: "endpoints" of future-directed <u>null</u> geodesics (celestial sphere × retarded time)

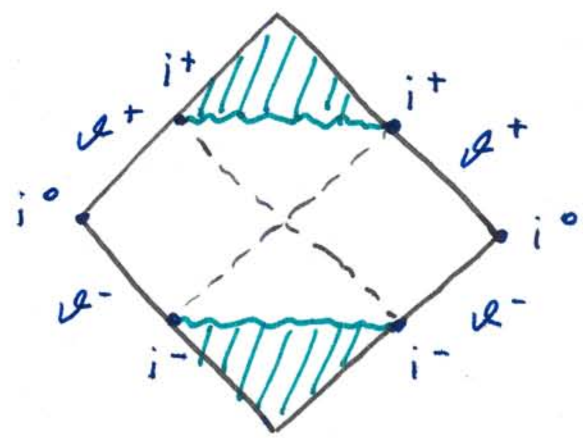
$\mathscr{I}^-$: "endpoints" of past-directed null geodesics (advanced time)

i^0: <u>one-point</u> compactification of space. (space-like geodesics)

Conformal Structure of Schwarzschild

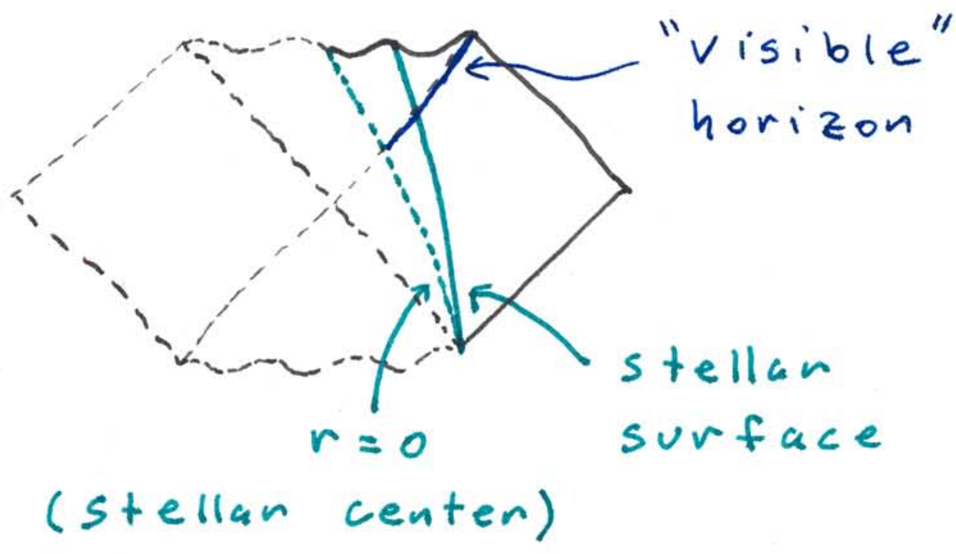
The Kruskal extension conformally embeds the tr-plane of the Schwarzschild spacetime into two-dimensional Minkowski spacetime. The Penrose diagram for Schwarzschild is therefore a subset of that for Minkowski:

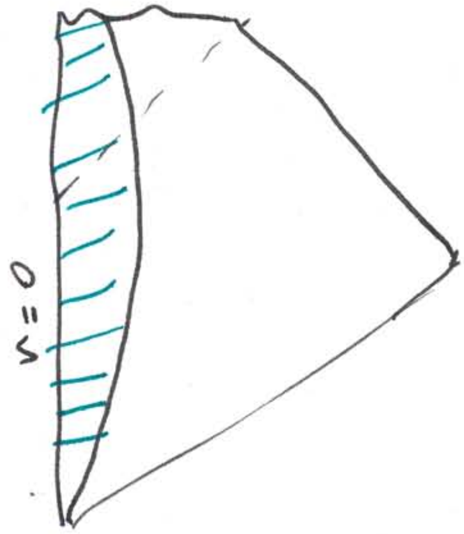




Physical Black Holes

White holes, like the one in the Kruskal solution, are not observed in Nature. Real black holes likely form from stellar collapse and they are (a) not eternal, (b) non-static and (c) non-spherical.



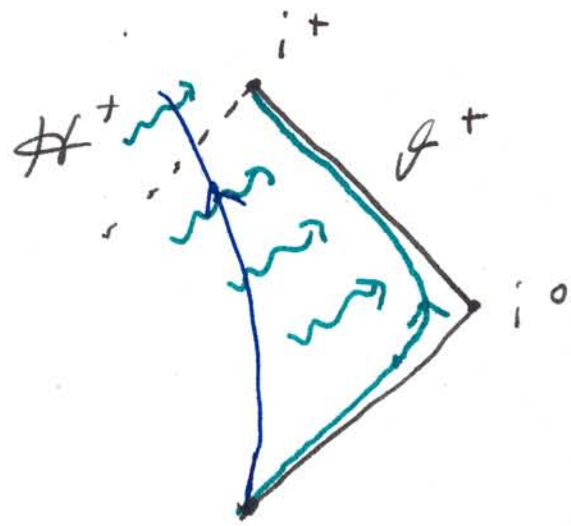

s = 0

A spacetime (M, g_{ab}) is <u>asymptotically flat</u> if it can be embedded conformally in a larger spacetime $(\tilde{M}, \hat{g}_{ab})$ with boundary

1) $\left.\overline{M}\right|_{\bar{g}} = \tilde{M}$ $\qquad g_{ab} = \omega^{-2}\,\hat{g}_{ab}$

2) $\omega \to 0$, $d\omega \neq 0$ on the boundary

3) $R_{ab} = 0$ "near" the boundary.



A asymptotically flat spacetime contains a black hole if there are points that are <u>outside</u> the causal past of $\mathcal{I}^+$.

$$\cancel{\tfrac{1}{2}} E^2 = \cancel{\tfrac{1}{2}} \cancelto{0}{\dot{r}^2} + \cancel{\tfrac{1}{2}} \left(1 - \frac{2M}{r}\right)\left(\frac{L^2}{r^2} + 1\right)$$

Circular orbit $\rightsquigarrow$ $\dot{r} = 0$

$$\frac{\partial V_{eff}}{\partial r} = 0$$

$$\Rightarrow M r^2 - L^2 r + 3 M L^2 = 0$$

$$\hat{V}_o = \frac{\partial}{\partial \tau} = \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t} + \frac{\partial \phi}{\partial \tau}\frac{\partial}{\partial \phi}$$

$$= \frac{E}{1 - \frac{2M}{r}}\frac{\partial}{\partial t} + \frac{L}{r^2}\frac{\partial}{\partial \phi}$$

$$\hat{V}_s = \left(1 - \frac{2M}{r}\right)^{-1/2}\frac{\partial}{\partial t}$$

$$ds^2 = -\left(\frac{\Delta - a^2 \sin^2\theta}{\Sigma}\right) dt^2 \qquad a = \frac{J}{M}$$

$$- 2\,\frac{a \sin^2\theta \,(r^2 + a^2 - \Delta)}{\Sigma}\, dt\, d\phi$$

$$+ \frac{\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2$$

$$+ \frac{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta\, d\phi^2$$

$$\Sigma = r^2 + a^2 \cos^2\theta \qquad \Delta = r^2 + a^2 - 2Mr$$