Lecture 19

The Interior of the Schwarzschild Black Hole

Global Structure of Spacetimes

Consider the metric

$$ds^2 = x^{-4}\,dx^2 + dy^2$$

on $\mathbb{R}^2$ with $x > 0$. It seems to be singular in the limit $x \to 0$, but we can write

$$ds^2 = (-d\,x^{-1})^2 + dy^2$$

$$= d\tilde{x}^2 + dy^2 \qquad (\tilde{x} := x^{-1})$$

This metric is Euclidean, and certainly not singular anywhere.

- $x \to 0 \quad \Rightarrow \quad \tilde{x} \to \infty$

$\Rightarrow$ $x = 0$ is at finite <u>coordinate</u> distance, but infinite <u>proper</u> distance. Physically, these are points "at infinity."

Now consider the metric

$$ds^2 = x^2 dx^2 + dy^2$$

This metric is degenerate at $x = 0$, so its inverse is singular there. But,

$$ds^2 = \left(\tfrac{1}{2} dx^2\right)^2 + dy^2$$

$$= d\tilde{x}^2 + dy^2 \qquad (\tilde{x} = \tfrac{1}{2}x^2)$$

Once again, there is no real singularity at $x = 0 \iff \tilde{x} = 0$. The metric is flat, and can be <u>extended</u> to negative values of $\tilde{x}$!

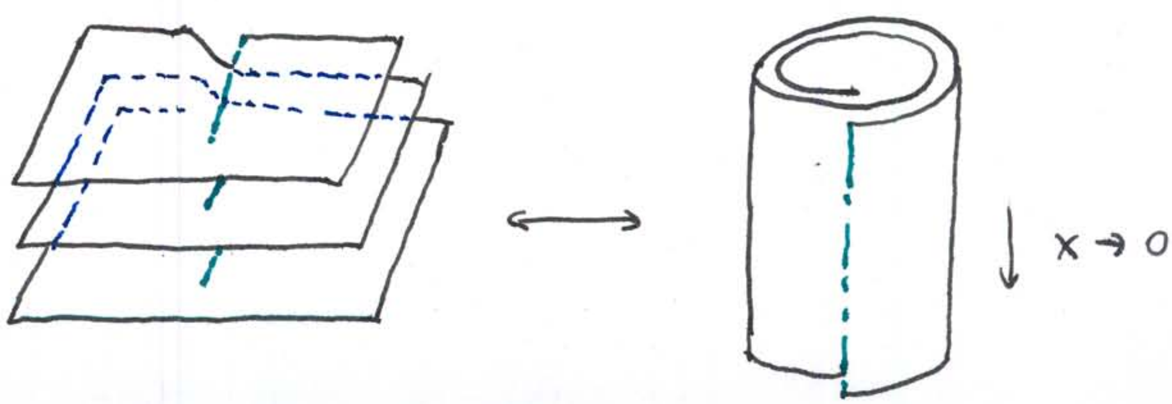
Now consider the metric

$$ds^2 = dx^2 + x^2 dy^2$$

This is again degenerate at $x = 0$, but we cannot simply reparameterize one of the coordinates this time. However, we can write

$$ds^2 = x^2 [x^{-2} dx^2 + dy^2]$$
$$= x^2 [d(\ln x)^2 + dy^2]$$

conformal factor



Conformal Transformations

Let M be a spacetime with a physical metric g_{ab} that is related to an unphysical (conformal) metric $\mathring{g}_{ab}$ by

$$g_{ab} = \Omega^2 \mathring{g}_{ab}$$

with Ω a scalar function on spacetime.

- Light cones are identical.
- Geometric quantities involving derivatives (connection, geodesics, curvature) will generally be different.

The metric connections ∇_a and $\mathring{\nabla}_a$ will be related by a tensor $C_{ab}{}^c$ given by

$$0 = \nabla_a g_{bc} = \mathring{\nabla}_a g_{bc} + C_{ab}{}^m g_{mc} + C_{ac}{}^m g_{bm}$$

$$= \mathring{\nabla}_a (\Omega^2 \mathring{g}_{bc}) + \Omega^2 (C_{ab}{}^m \mathring{g}_{mc} + C_{ac}{}^m \mathring{g}_{bm})$$

$$= \mathring{g}_{bc} \mathring{\nabla}_a \Omega^2 + \Omega^2 \cdot 2 C_{a(bc)}$$

Here, we have lowered the index on $C_{ab}{}^c$ using the unphysical conformal metric $\mathring{g}_{ab}$.

Note that we also have

$$C_{[ab]}{}^c = 0$$

because both ∇_a and $\mathring{\nabla}_a$ are torsion-free

This lets us calculate $C_{ab}{}^{c}$ in the familiar way:

$$C_{abc} = -\mathring{g}_{bc} \mathring{\nabla}_a \ln \Omega^2 - C_{acb}$$

$$= -\mathring{g}_{bc} \mathring{\nabla}_a \ln \Omega^2 - C_{cab}$$

$$= -\mathring{g}_{bc} \mathring{\nabla}_a \ln \Omega^2 - \left(-\mathring{g}_{ab} \mathring{\nabla}_c \ln \Omega^2 - C_{bca}\right)$$

$$= -\mathring{g}_{bc} \mathring{\nabla}_a \ln \Omega^2 + \mathring{g}_{ab} \mathring{\nabla}_c \ln \Omega^2 - \mathring{g}_{ca} \mathring{\nabla}_b \ln \Omega^2 - C_{abc}$$

$$\Rightarrow \quad C_{ab}{}^{c} = \mathring{g}_{ab} \mathring{\nabla}^c \ln \Omega - 2\,\delta^c_{(a} \mathring{\nabla}_{b)} \ln \Omega$$

Note that we have raised the index again here using the unphysical metric $\mathring{g}^{ab} = \Omega^2 g^{ab}$.

$$\left(\mathring{g}_{ab} = \Omega^{-2} g_{ab}\right)$$

Null Geodesics

Null geodesics have the peculiar property that they remain invariant under a conformal transformation of spacetime.

Let $\mathring{U}^a$ denote the tangent to an affinely-parameterized geodesic of the unphysical metric $\mathring{g}_{ab}$. Then,

$$\mathring{U}^a \nabla_a \mathring{U}^c = \mathring{U}^a \mathring{\nabla}_a \mathring{U}^c - \mathring{U}^a C_{ab}{}^c \mathring{U}^b$$
$$= -\mathring{U}^a \mathring{U}^b \left(\mathring{g}_{ab} \mathring{\nabla}^c \ln \Omega - 2 \delta^c_{(a} \mathring{\nabla}_{b)} \ln \Omega \right)$$

The first term, which generally would give an acceleration in the physical metric, vanishes if and only if $\mathring{U}^a$ is null.

The second term does not vanish, but shows only that $\mathring{U}^a$ is not <u>affinely</u> parameterized in the physical metric:

$$\mathring{U}^a \nabla_a \mathring{U}^c = 2 \mathring{U}^c \mathring{U}^a \mathring{\nabla}_a \ln \Omega$$

$$= \frac{\mathring{U}^c}{\Omega^2} \mathring{U}^a \mathring{\nabla}_a \Omega^2$$

$$\Rightarrow \frac{\mathring{U}^a}{\Omega^2} \cdot \frac{1}{\Omega^2} \nabla_a \mathring{U}^c - \frac{\mathring{U}^a}{\Omega^2} \cdot \mathring{U}^c \frac{\mathring{\nabla}_a \Omega^2}{\Omega^4} = 0$$

$$= \frac{\mathring{U}^a}{\Omega^2} \nabla_a \left(\frac{\mathring{U}^c}{\Omega^2} \right)$$

Thus, the affine parameterization in the <u>physical</u> metric g_{ab} has tangent $U^a := \mathring{U}^a / \Omega^2$.

$$\frac{d}{d\lambda} = \frac{1}{\Omega^2} \frac{d}{d\mathring{\lambda}} \rightsquigarrow d\lambda = \Omega^2 d\mathring{\lambda}$$

The Rindler Wedge

We have found previously that the flat Minkowski metric takes the form

$$ds^2 = e^{2g\xi}\left(-d\tau^2 + d\xi^2\right)$$

in the radio-coordinates of an accelerating observer.

Introduce the metric spatial coordinate

$$x(\xi) = \int_0^\xi ds = \int_0^\xi e^{g\xi}\, d\xi$$

$$= \frac{1}{g}\left(e^{g\xi} - 1\right)$$

$$\Rightarrow \quad \xi(x) = \frac{1}{g}\ln\left(1 + g x\right)$$

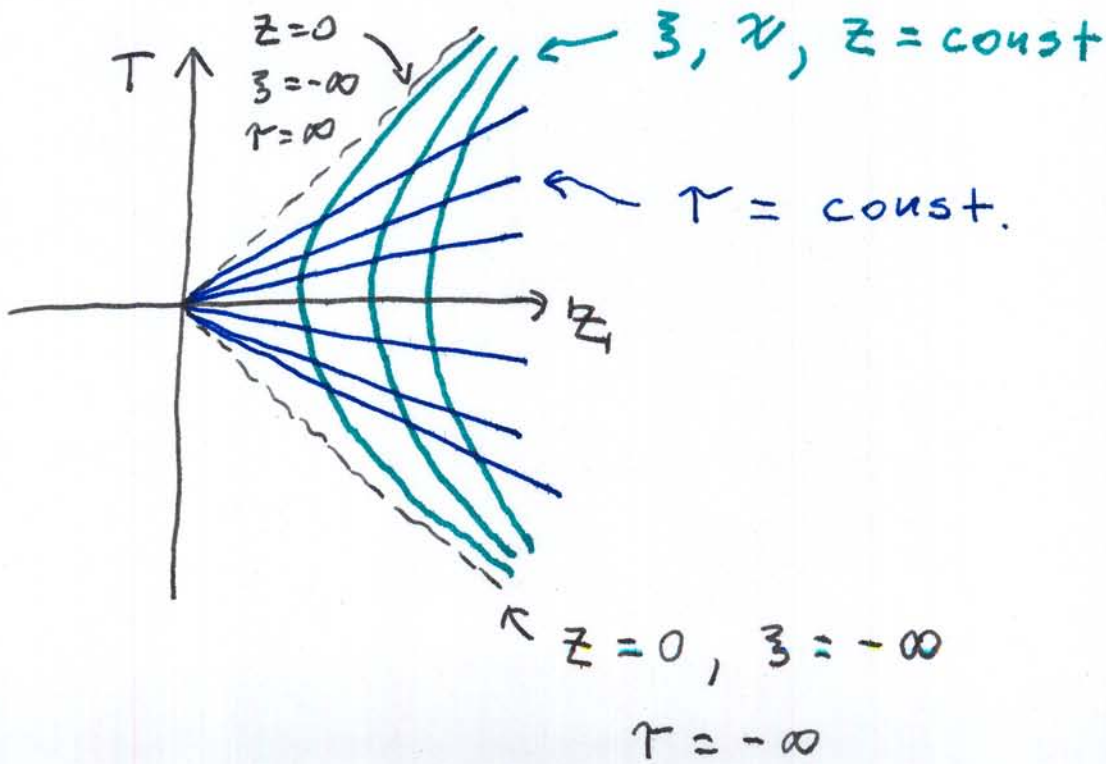
Thus, in metric coordinates,

$$ds^2 = -(1+gx)^2\, d\tau^2 + dx^2$$

Define $z := x + g^{-1}$ and we find the <u>Rindler metric</u>

$$ds^2 = -(gz)^2\, d\tau^2 + dz^2$$

This metric has some properties similar to the Schwarzschild metric.



Now, suppose we know <u>only</u> the Rindler metric

$$ds^2 = -g^2 z^2 dt^2 + dz^2$$

on the region $z > 0$. How do we recover the full Minkowski spacetime?

1) Note that the Rindler metric is conformally flat:

$$ds^2 = (gz)^2 \left[-dt^2 + \frac{dz^2}{g^2 z^2} \right]$$

$$= (gz)^2 \left[-dt^2 + \left(d\,\frac{\ln gz}{g} \right)^2 \right]$$

$\Omega = gz$ $\qquad$ $\overset{\circ}{ds}{}^2 = -dt^2 + d\xi^2$

$\xi := g^{-1} \ln(gz)$

Note: $z \to 0 \iff \xi \to -\infty$

2) Note that the singular points at $z = 0$ are at <u>finite</u> affine parameter along a null geodesic:

$$d\lambda = \Omega^2 \, d\mathring{\lambda} = -g^2 z^2 \, d\xi$$

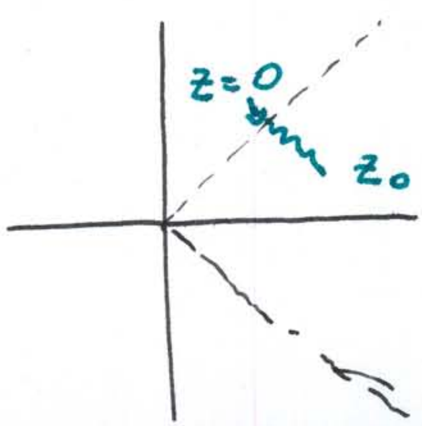
$dt = -d\xi$ affine in <u>flat</u> space

$$= -e^{2g\xi} \, d\xi$$

$$\leadsto \Delta\lambda = \int_{\xi_0}^{-\infty} -e^{2g\xi} \, d\xi$$

$$= \frac{-1}{2g} e^{2g\xi} \Big/_{\xi_0}^{-\infty} = \frac{1}{2g} e^{2g\xi_0}$$

$$= \frac{1}{2} g z_0^2 \leftarrow \underline{\text{finite}}$$



A light ray reaches the "singularity" in a <u>finite</u> time.

3) Study the metric in <u>null coordinates</u> (u, v) that are adapted to the geodesics

$$d\mathring{s}^2 = -dt^2 + d\zeta^2 = -du\,dv$$

$$u = t - \zeta \qquad v = t + \zeta$$

The physical metric is

$$ds^2 = (gz)^2\, d\mathring{s}^2 = e^{2g\zeta} \cdot -du\,dv$$

$$= -e^{g(v-u)}\,du\,dv$$

$$= -\,d\big(\underbrace{-g^{-1}e^{-gu}}_{U}\big)\, d\big(\underbrace{g^{-1}e^{gv}}_{V}\big)$$

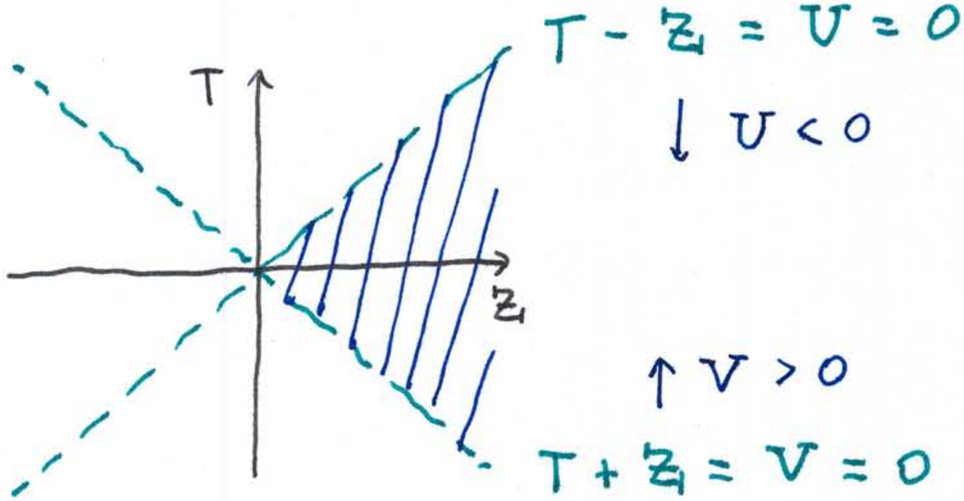
Note that

$$\begin{array}{c} -\infty < u < \infty \\ -\infty < v < \infty \end{array} \quad \rightsquigarrow \quad \begin{array}{c} -\infty < U < 0 \\ 0 < V < \infty \end{array}$$

We have, of course, just found the inertial coordinates of the original Minkowski metric:

<u>Define</u>:
$$T := \tfrac{1}{2}(U + V)$$
$$Z_1 := \tfrac{1}{2}(V - U)$$

$$ds^2 = -dU\,dV = -dT^2 + dZ_1^2$$



The <u>physical</u> metric can be extended through the "surface" $z = 0$ to recover all of the Minkowski spacetime.