Lecture 18

The Schwarzschild Black Hole

What's going on at $r = 2M$??

Perihelion Precession

An orbit <u>near</u> the stable equilibrium at $r = R_+$ will oscillate in the effective potential with frequency

$$\omega^2 \equiv \frac{\partial^2 V_{eff}}{\partial r^2}(R_+)$$

$$= -2\frac{M}{R_+^3} + 3\frac{L^2}{R_+^4} - 4\frac{3ML^2}{R_+^5}$$

$$= \frac{-2MR_+^2 + (R_+ - 4M)3L^2}{R_+^5}$$

Recall that

$$MR_+^2 - L^2 R_+ + 3ML^2 = 0$$

$$\Rightarrow \quad L^2 = \frac{MR_+^2}{R_+ - 3M}$$

$$\Rightarrow \quad \omega^2 = \frac{M}{R_+^3}\,\frac{\overbrace{3(R_+ - 4M) - 2(R_+ - 3M)}^{(R_+ - 6M)}}{(R_+ - 3M)}$$

Contrast this with the Newtonian result:

$$\mathring{\omega}^2 \cong \frac{\partial^2 V_{eff}}{\partial r^2}(R_0) \qquad L^2 = M R_0$$

$$= -2\frac{M}{R_0^3} + 3\frac{L^2}{R_0^4} = \frac{M}{R_0^3} = \frac{L^2}{R_0^4}$$

Recall that $L := r^2\dot{\phi}$, so $\mathring{\omega}^2 = \dot{\phi}^2$ in Newtonian gravity. This is why elliptic orbits close. In relativity, however,

$$\omega^2 = \frac{L^2}{R_+^5}(R_+ - 6M) = \left(1 - \frac{6M}{R_+}\right)\dot{\phi}^2$$

$$\Rightarrow \omega = \sqrt{1 - \frac{6M}{R_+}}\,\dot{\phi} \cong \underline{\left(1 - \frac{3M}{R_+}\right)\dot{\phi}}$$

Thus, nearly circular orbits <u>precess</u> in general relativity.

Define the precession frequency

$$\omega_p := \dot\phi - \omega \cong \frac{3M}{R_+}\dot\phi = \frac{3M}{R_+}\frac{L}{R_+^2}$$

$$= \frac{3M}{R_+^3}\sqrt{\frac{MR_+^2}{R_+ - 3M}} \cong \frac{3M^{3/2}}{R_+^{5/2}}$$

We can can calculate this precession rate for Mercury:

$$\omega_p = \frac{3M^{3/2}}{R^{5/2}} = \frac{3c}{R^{5/2}}\left(\frac{GM}{c^2}\right)^{3/2}$$

$$= \frac{3(6.67\times10^{-8})^{3/2}(1.99\times10^{33})^{3/2}}{(5.79\times10^{12})^{5/2}(3.00\times10^{10})^2}$$

$$= 6.32 \times 10^{-14}\ \frac{\text{rad}}{\text{sec}}$$

$$= 1.30 \times 10^{-8}\ \frac{''}{\text{sec}}$$

$$= 41.1\ \frac{''}{\text{century}}$$ $\leftarrow$ observed precession!

General Orbits

We have so far considered only quasi-circular orbits. More generally, we must use the full geodesic equation. As in Newtonian theory, it is mathematically convenient to

(a) introduce the inverse radial coordinate $u := r^{-1}$

(b) reparameterize the curve using a spatial coordinate
- ϕ for <u>bound</u> orbits
- u for unbound

and solve for the locus of points in space.

The effective "Newtonian energy," with dimensional constants restored, becomes

$$\frac{1}{2}\frac{c^2}{L^2}(E^2+2K) = \frac{1}{2}\left(\frac{du}{d\phi}\right)^2 + \frac{1}{2}u^2 + 2K\frac{GM}{L^2}u - \frac{GM}{c^2}u^3$$

For bound orbits, this gives the geodesic equation

$$\frac{d^2u}{d\phi^2} + u = -2K\frac{GM}{L^2} + 3\frac{GM}{c^2}u^2$$

For unbound orbits, it is better to keep the first integral

$$\frac{d\phi}{du} = \left[b^{-2} - u^2 - 4K\frac{GM}{L^2}u + 2\frac{GM}{c^2}u^3\right]^{-1/2}$$

$b = \frac{L}{c}(E^2+2K)^{-1/2}$ = impact parameter at infinity.

Null Geodesics

For null geodesics, the effective potential has $2K = 0$, so

$$V_{eff}(r) = \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

- The Newtonian term from the time-like case is missing, which is "why" Newtonian gravity does not affect light.
- The centrifugal term is the same because light moves on straight lines.
- The relativistic term lets gravity act on light rays.

In[124]:=

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Clear[L]
```

In[125]:=

```
Vn = -M / r + L^2 / (2 * r^2)
Ve = Vn - M * L^2 / r^3
Vl = Ve + M / r
Vc = Vn + M / r
```

Out[125]=

$$\frac{L^2}{2\,r^2} - \frac{M}{r}$$

Out[126]=

$$-\frac{L^2\,M}{r^3} + \frac{L^2}{2\,r^2} - \frac{M}{r}$$

Out[127]=

$$-\frac{L^2\,M}{r^3} + \frac{L^2}{2\,r^2}$$

Out[128]=

$$\frac{L^2}{2\,r^2}$$

In[129]:=

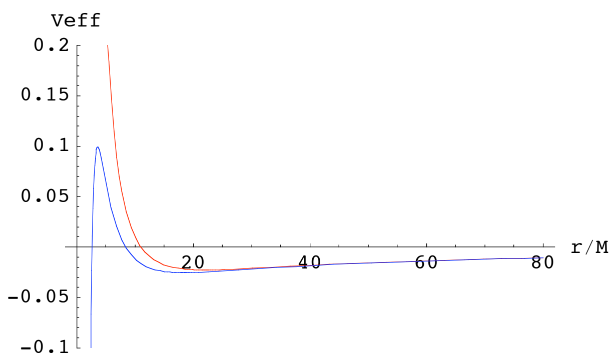
```
L = Sqrt[22] * M
```

Out[129]=

$$\sqrt{22}\ M$$

In[130]:=

```
Plot[Evaluate[{Vn, Ve} /. M → 1], {r, 1, 80}, AxesLabel → {"r/M", "Veff"},
 PlotStyle → {{RGBColor[1, 0, 0]}, {RGBColor[0, 0, 1]}}, PlotRange → {-0.1, 0.2}]
```

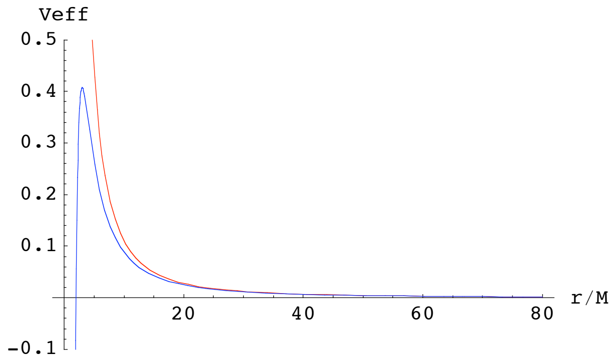


Out[130]=

- Graphics -

In[132]:=

```
Plot[Evaluate[{Vc, Vl} /. M → 1], {r, 1, 80}, AxesLabel → {"r/M", "Veff"},
 PlotStyle → {{RGBColor[1, 0, 0]}, {RGBColor[0, 0, 1]}}, PlotRange → {-0.1, 0.5}]
```



Out[132]=

- Graphics -

In[133]:=

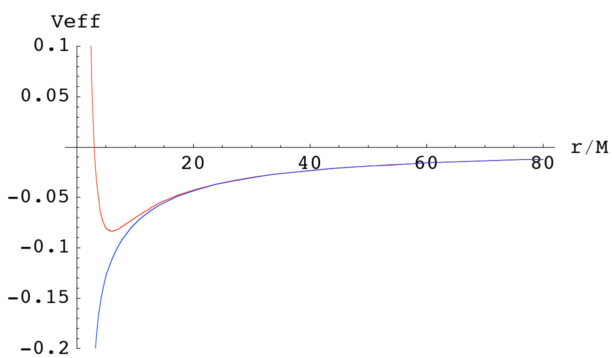
```
L = Sqrt[6] * M
```

Out[133]=

$\sqrt{6}$ M

In[135]:=

```
Plot[Evaluate[{Vn, Ve} /. M → 1], {r, 1, 80}, AxesLabel → {"r/M", "Veff"},
 PlotStyle → {{RGBColor[1, 0, 0]}, {RGBColor[0, 0, 1]}}, PlotRange → {-0.2, 0.1}]
```

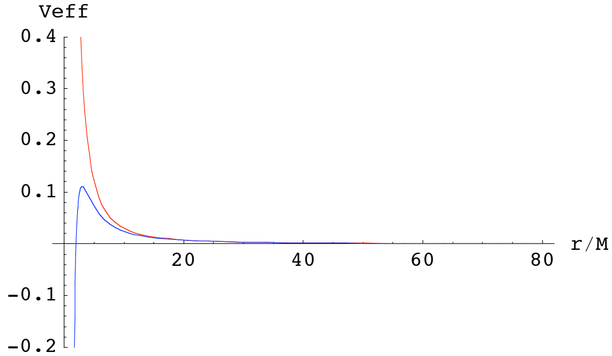


Out[135]=

- Graphics -

In[138]:=

```
Plot[Evaluate[{Vc, Vl} /. M → 1], {r, 1, 80}, AxesLabel → {"r/M", "Veff"},
 PlotStyle → {{RGBColor[1, 0, 0]}, {RGBColor[0, 0, 1]}}, PlotRange → {-0.2, 0.4}]
```



Out[138]=

```
- Graphics -
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Radial Geodesics

When $L = r^2 \dot{\phi} = 0$, the effective "Newtonian energy" becomes simply

$$\frac{1}{2} E^2 = \frac{1}{2} \dot{r}^2 - K\left(1 - \frac{2M}{r}\right)$$

$$\rightsquigarrow \frac{1}{2}\left(E^2 + 2K\right) = \frac{1}{2}\dot{r}^2 + 2K\frac{M}{r}$$

For time-like geodesics, the right side has exactly the Newtonian form!

But there are relativistic effects because we are in proper time parameterization

$$E = \left(1 - \frac{2M}{r}\right)\dot{t} = \text{const.}$$

Escape Velocity

What minimum speed must an outward-bound test particle have at $r = R$ to escape to infinity? $(2K = -1)$

$$\frac{1}{2}(E^2 - 1) = \frac{1}{2}\dot{r}^2 - \frac{M}{r} \to 0$$

$$\Rightarrow \quad E = \left(1 - \frac{2M}{r}\right)\dot{t} = 1$$

The four-velocity, in static coordinates, at $r = R$ is

$$\hat{v}_e = \frac{\partial}{\partial \tau} = \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t} + \frac{\partial r}{\partial \tau}\frac{\partial}{\partial r}$$

$$= \left(1 - \frac{2M}{R}\right)^{-1}\frac{\partial}{\partial t} + \sqrt{\frac{2M}{R}}\,\frac{\partial}{\partial r}$$

We can check

$$\hat{v}_e \cdot \hat{v}_e = -\left(1 - \frac{2M}{R}\right)\cdot\left(1 - \frac{2M}{R}\right)^{-2}$$

$$+ \left(1 - \frac{2M}{R}\right)^{-1} \cdot \frac{2M}{R} = -1$$

In the limit as $R \to 2M$, the escape velocity four-vector becomes undefined. Why?

What speed does a static observer at $r = R$ observe the escaping particle to have?

$$\hat{V}_s(R) = C\frac{\partial}{\partial t} = \left(1 - \frac{2M}{R}\right)^{-1/2}\frac{\partial}{\partial t}$$

$$\hat{V}_s(R) \cdot \hat{V}_s(R) = -\left(1 - \frac{2M}{R}\right)c^2 = -1$$

In special relativity, the product of 4-velocities is

$$\hat{V}_1 \cdot \hat{V}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \gamma \\ \gamma\vec{v} \end{pmatrix} = -\gamma = -(1 - v^2)^{-1/2}$$

$$\rightsquigarrow \hat{V}_s \cdot \hat{V}_e = -\left(1 - \frac{2M}{R}\right)\left(1 - \frac{2M}{R}\right)^{-1/2}\left(1 - \frac{2M}{R}\right)^{-1}$$

$$\Rightarrow -(1 - v_e^2)^{-1/2} = -\left(1 - \frac{2M}{R}\right)^{-1/2}$$

$$v_e^2 = \frac{2M}{R} \quad \leftarrow \text{Newtonian result!}$$

$\to c^2$ at $R = 2M$

So, the local static observers see the particle move with speed $v \to 1$ as $R \to 2M$. Note that a static observer at infinity moves with proper time $\tau_\infty = t$ and

$$\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = \frac{\sqrt{\frac{2M}{R}}}{\left(1-\frac{2M}{R}\right)^{-1}} = \left(1-\frac{2M}{R}\right)\sqrt{\frac{2M}{R}}$$

goes to zero as $R \to 2M$.

Indeed, if we parameterize the geodesic with coordinate time t instead of proper time τ, we have

$$\overset{\infty}{V}_e = \frac{\partial t}{\partial t}\frac{\partial}{\partial t} + \frac{\partial r}{\partial t}\frac{\partial}{\partial r}$$

null as $R \to 2M$

$$= \frac{\partial}{\partial t} + \left(1-\frac{2M}{R}\right)\sqrt{\frac{2M}{R}}\frac{\partial}{\partial r}$$

Radial Infall

Suppose the static observer at $r = R$ releases a test particle at some time.

$$\hat{V}_0 = \hat{V}_s(R)$$

We can integrate the "energy" equation to find

$$\frac{1}{2}(E^2 - 1) = \frac{1}{2}\dot{r}^2 - \frac{M}{r} = -\frac{M}{R}$$

$$\Rightarrow \frac{dr}{d\tau} = -\sqrt{\frac{2M}{r} - \frac{2M}{R}}$$

$$\Rightarrow \tau = -\int_R^r \frac{ds}{\sqrt{R/s - 1}} \cdot \sqrt{\frac{R}{2M}}$$

$$= \sqrt{\frac{R}{2M}}\left(\sqrt{r(R-r)} + R\cos^{-1}\sqrt{\frac{r}{R}}\right)$$

Thus, τ remains <u>finite</u> as the particle falls to $r \to 2M$.

However, we have

$$\frac{dt}{dr} = \frac{\dot{t}}{\dot{r}} = \frac{E}{1 - \frac{2M}{r}} \cdot \frac{-1}{\sqrt{\frac{2M}{r} - \frac{2M}{R}}}$$

$$= \frac{-r\sqrt{R-2M}}{\underbrace{(r-2M)}\sqrt{2M(R-r)}}$$

The new term in the denominator causes a logarithmic <u>divergence</u> in the integral as $r \to 2M$.

Radial infall takes finite <u>proper</u> time τ but infinite <u>coordinate</u> time t to reach the limit $r \to 2M$.

Curvature of Schwarzschild

Schwarzschild spacetime is a vacuum solution, so it is Einstein- and, therefore, Ricci-flat ($R_{ab} = 0$). But it is not (Riemann) flat:

$$(FN')' \, dt \wedge dr = \frac{F}{N} (FN')' e^t \wedge e^r$$
$$= \frac{1}{2} (N^2)'' e^t \wedge e^r = -\frac{2M}{r^3} e^t \wedge e^r$$

We can calculate other terms in the Riemann matrix similarly to find

$$R_\alpha{}^\beta = \frac{M}{r^3} \begin{pmatrix} 0 & -2e^t\wedge e^r & e^t \wedge e^\theta & e^t\wedge e^\phi \\ -2e^t\wedge e^r & 0 & -e^r\wedge e^\theta & -e^r \wedge e^\phi \\ e^t\wedge e^\theta & e^r \wedge e^\theta & 0 & 2e^\theta\wedge e^\phi \\ e^t \wedge e^\phi & e^r\wedge e^\phi & -2e^\theta \wedge e^\phi & 0 \end{pmatrix}$$

This certainly doesn't <u>look</u> like it has a problem as $r \to 2M$, but we have used a basis based on coordinates with curious properties there.

A standard test to look for <u>true</u> singularities in the field is to construct <u>scalar</u> <u>curvature invariants</u> like

$$
\begin{aligned}
I &:= R_{abcd} R^{abcd} \\
&= R_{ab\alpha\beta} R^{ab\alpha\beta} \\
&= -R_{ab\alpha}{}^{\beta} R^{ab}{}_{\beta}{}^{\alpha} \\
&= -2\left(\frac{M}{r^3}\right)^2 \Big[(-2e^t \wedge e^r)^2 + (e^t \wedge e^\theta)^2 \\
&\qquad + (e^t \wedge e^\phi)^2 - (e^r \wedge e^\theta)^2 \\
&\qquad - (e^r \wedge e^\phi)^2 - (-2e^\theta \wedge e^\phi)^2 \Big]
\end{aligned}
$$

Note that we have

$$(e^t{}_{\wedge} e^r)^2 := g^{am} g^{bn} \cdot 2 e^t_{[a} e^r_{b]} \cdot 2 e^t_{[m} e^r_{n]}$$

$$= 2 e^t_{[a} e^r_{b]} \cdot -2 e^a_t e^b_r$$

$$= -2$$

Following through, we find

$$I = 4 \frac{M^2}{r^6} [4 + 1 + 1 + 1 + 1 + 4]$$

$$= \frac{48 M^2}{r^6}$$

This is <u>certainly</u> not singular as $R \to 2M$. In fact, no curvature invariant is singular there. The apparent singularity in the metric results from the poor behavior of t in the limit!