Lecture 13

Structure of the Einstein Field Equations

Contracted Bianchi Identity

We previously contracted the Bianchi identities to find:

$$R_{[ab]} = \frac{1}{2} R_{abc}{}^{c}$$

$$\nabla_{[a} R_{b]c} = -\frac{1}{2} \nabla_d R_{abc}{}^{d}$$

For the metric connection, we find the new anti-symmetry

$$\left(R_{abcd} = -R_{abdc}\right) g^{cd}$$

$$\Rightarrow R_{abc}{}^{c} = -R_{abd}{}^{d} = 0$$

$$\Rightarrow R_{ab} = R_{ba}$$

We now contract the second Bianchi identity one more time:

$$\left(\nabla_a R_{bc} - \nabla_b R_{ac} + \nabla_d R_{abc}{}^{d} = 0\right) g^{ac}$$

The result is

$$\nabla_a R_b{}^a - \nabla_b R + \nabla_d R_{ab}{}^{ad} = 0$$

$$R := R_{ab} g^{ab}$$ <u>scalar curvature</u>

(<u>Ricci scalar</u>)

$$\nabla_d R_{ab}{}^{ad} = \nabla_d R_{ba}{}^{da}$$

$$= \nabla_d R_b{}^d = \nabla_a R_b{}^a$$

$$\Rightarrow 2 \nabla_a R_b{}^a - \nabla_b R = 0$$

$$= 2 \nabla_a \left(\underbrace{R_b{}^a - \tfrac{1}{2} R \delta_b^a}_{} \right)$$

$$G_{ab} := R_{ab} - \tfrac{1}{2} R g_{ab}$$

<u>Einstein tensor</u>

The Einstein tensor computed from the curvature of the metric connection is divergence-free in that connection.

Einstein Field Equation

$$G_{ab} = 8\pi G \, T_{ab}$$

- G_{ab} = Einstein curvature
- 8π set by Newtonian limit
- G = Newton's constant (no new "tunable" parameter)
- T_{ab} = stress-energy of matter sources.

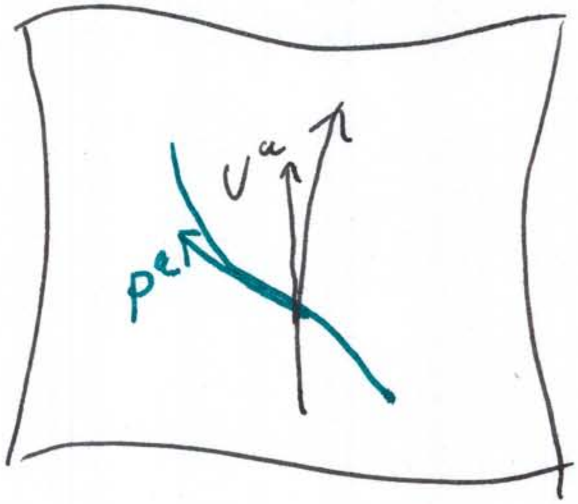
It is impossible to solve the field equations if energy is not conserved:

$$\nabla_a \left(G^{ab} = 8\pi \, T^{ab} \right) \qquad \begin{pmatrix} G = 1 \\ c = 1 \end{pmatrix}$$

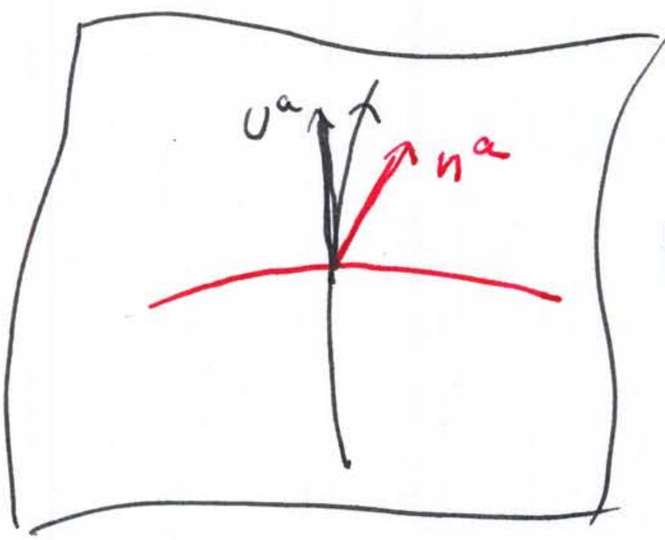
$$0 = 8\pi \underbrace{\nabla_a T^{ab}}$$

conservation of energy.

$$R_{ab} = \emptyset\, T_{ab}$$



$$E = -u^a p_a$$



$$E = -u^a n^b T_{ab}$$

$$T_{ab} = \frac{1}{4\pi}\left(F_a{}^m F_b{}^n - \frac{1}{2} g_{ab} F_{mn} F^{mn} \right)$$

Analogy to Maxwell

$$\nabla_{[a} F_{bc]} = 0 \qquad \nabla_a F^{ab} = -4\pi j^b$$

- F_{ab} = Maxwell field tensor.
- j^a = charge-current vector of charged sources.

Take the divergence of the inhomogeneous equation:

$$\nabla_{[a} \nabla_{b]} F^{ab} = -4\pi \nabla_b j^b$$

$$= -\tfrac{1}{2} R_{abc}{}^{a} F^{cb} - \tfrac{1}{2} R_{abc}{}^{b} F^{ac}$$

$$= \tfrac{1}{2} R_{bc} F^{cb} - \tfrac{1}{2} R_{ac} F^{ac}$$

$$= 0 \quad \Rightarrow \quad \nabla_b j^b = 0$$

It is <u>impossible</u> to solve the Maxwell equations if <u>charge</u> is <u>not</u> conserved.

Charge conservation is related to the <u>gauge invariance</u> of Maxwell theory:

$$\nabla_{[a} F_{bc]} = 0 \quad \Rightarrow \quad F_{bc} = 2\nabla_{[b} A_{c]}$$

(locally)

$$\Rightarrow \nabla_a \left(\nabla^a A^b - \nabla^b A^a \right) = -4\pi j^b$$

$$= \nabla_a \nabla^a A^b + R_a{}^b{}_c{}^a A^c - \nabla^b \nabla_a A^a$$

$$= \underbrace{\nabla_a \nabla^a A^b - R^b{}_c A^c}_{\text{vector wave operator}} - \nabla^b \nabla_a A^a$$

vector wave operator

$\Box A^b \quad \left(\nabla_b \Box A^b = \Box \nabla_b A^b \right)$

Two things to note:

- $\left(\Box \delta^b_a - \nabla^b \nabla_a \right) \nabla^a \psi = 0$
- $\nabla_b \underbrace{\left(\Box \delta^b_a - \nabla^b \nabla_a \right)}_{\text{Maxwell operator } \mathcal{M}} A^a = 0$

Maxwell operator $\mathcal{M}$

(<u>not</u> invertible)

The Maxwell equations are

- underdetermined because of gauge ambiguity
- overdetermined because of continuity constraint

What is the relation?

$$\langle \operatorname{grad} \psi, \mathcal{M} A \rangle_1 = - \langle \psi, \operatorname{div} \mathcal{M} A \rangle_0$$

$\langle \cdot, \cdot \rangle_{0,1}$ is the standard L^2 inner product on the space of scalar, vector fields.

- We have integrated by parts to show $\operatorname{grad}^+ = -\operatorname{div}$.
- The Maxwell operator $\mathcal{M}$ is self-adjoint in $\langle \cdot, \cdot \rangle_1$. (exercise)

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$\mathcal{M}$ self-adjoint

has Kernel

$\mathcal{M} A = j$

$$\underbrace{\langle \operatorname{grad} \psi, j \rangle_1}_{= 0} = - \langle \psi, \operatorname{div} j \rangle$$

How do we solve the Einstein field equations?

$$G_{ab} = 8\pi T_{ab}$$

In four spacetime dimensions, these are ten non-linear second-order differential equations for the ten metric component fields g_{ab}.

- overdetermined: $\nabla^a G_{ab} = 0$ for <u>all</u> metrics g_{ab}
- underdetermined:

$$\int 2 \nabla_{(a} X_{b)} \cdot G^{ab} \cdot \text{vol}$$

$$= -2 \int X_b \nabla_a G^{ab} \, \text{vol} = 0$$

$\rightsquigarrow$ "gauge" $\quad \dot{g}_{ab} = 2 \nabla_{(a} X_{b)}$

$$= \mathcal{L}_X g_{ab}$$

Viable Strategies

1) Symmetry Reduction

- Impose physically-motivated symmetries on g_{ab}.
- Calculate G_{ab} for an arbitrary symmetric g_{ab}.
- Solve the equations for a given, symmetric T_{ab}.

2) Perturbation Theory

- For weak sources in a given background, develop a series expansion for the real g_{ab} in a suitable small parameter.

3) Numerical Simulation

- Implement and solve the equations on a coordinate grid using supercomputers.

Linearized Gravity

Suppose the <u>physical metric</u> g_{ab} is "close" to a given <u>background metric</u> $\mathring{g}_{ab}$ in a region of spacetime.

If $\mathring{g}_{ab}$ satisfies $\mathring{G}_{ab} = 0$, what equations must

$$\delta g_{ab} := g_{ab} - \mathring{g}_{ab}$$

satisfy so that $G_{ab} = 8\pi T_{ab}$ to leading order?

To formulate the problem mathematically, we assume there is a parameter λ in the problem and build a power series for $g_{ab}(\lambda)$.

⇝ What is $\dot{g}_{ab}(\lambda = 0)$?

We want to set

$$\dot{G}_{ab}(\lambda=0) = 8\pi \dot{T}_{ab}(\lambda=0).$$

To calculate $\dot{G}_{ab}$, we must first find $\dot{\nabla}_{ab}{}^{c}$, the first-order change in the connection.

For all λ: $\nabla_a g_{bc} = 0$

$$\Rightarrow \dot{\nabla}_{ab}{}^{m} g_{mc} + \dot{\nabla}_{ac}{}^{m} g_{bm} + \nabla_a \dot{g}_{bc} = 0$$

For all λ: $T_{ab}{}^{c} = 0$

$$\Rightarrow \dot{T}_{ab}{}^{c} = 0$$

$$\Rightarrow \dot{\nabla}_{[ab]}{}^{c} = 0$$

$$\Rightarrow \begin{cases} \dot{\nabla}_{abc} + \dot{\nabla}_{acb} = -\nabla_a \dot{g}_{bc} \\ \dot{\nabla}_{abc} - \dot{\nabla}_{bac} = 0 \end{cases}$$

$$\Rightarrow \dot{\nabla}_{ab}{}^{c} = -\tfrac{1}{2}\mathring{g}^{cd}\left(2\mathring{\nabla}_{(a}\dot{g}_{b)d} - \mathring{\nabla}_d \dot{g}_{ab}\right)$$