Lecture 10

Torsion and Curvature

Parallel Transport

To do differential calculus with vector fields, we must take derivatives of one vector field along another: $\nabla_V W^a$

The Lie derivative does <u>not</u> let us do this!

- $\mathcal{L}_V W^a = [V, W]^a$
- $[V, W](f) = V(W(f)) - W(V(f))$
- $V(W(f)) = V^\alpha \partial_\alpha (W^\beta \partial_\beta (f))$

$$= V^\alpha \partial_\alpha (W^\beta) \partial_\beta (f) + V^\alpha W^\beta \partial_\alpha \partial_\beta (f)$$

$$\Rightarrow [V, W](f) = [V(W^\beta) - W(V^\beta)] \partial_\beta (f) + 2 V^\alpha W^\beta \underbrace{\partial_{[\alpha} \partial_{\beta]}}_{=0} (f)$$

$\mathcal{L}_V W^a$ requires two vector <u>fields</u>

It does <u>not</u> let us take the derivative of a vector field W^a along a <u>single</u> integral curve of V^a.

We need an additional structure $\nabla_{\dot\gamma} W^a$ to take the derivative of $W^a(p)$ as we move along $\gamma(t)$.

This should take <u>derivatives</u> of W^a, but be <u>algebraic</u> in $\dot\gamma^a$.

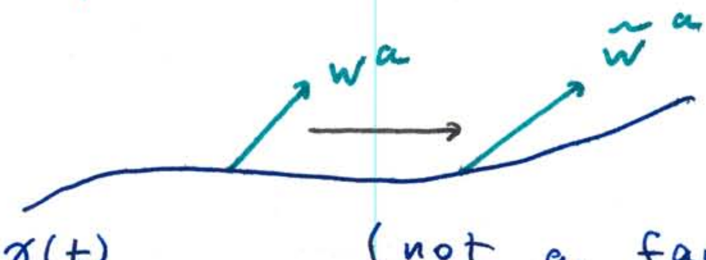
functionally linear

$$\nabla_{fV} W^a = f \nabla_V W^a$$

function

$$\nabla_V (f W^a) = f \nabla_V W^a + W^a \nabla_V f$$

Naturally, we define $\nabla_V f := V(f)$.



(<u>not</u> a <u>family</u> of curves)

Derivative Operators

(Covariant Derivative, $\nabla_W V^a$
Affine Connection)

Given: vector W^a and
vector field V^a

- $\nabla_W V^a$ is functionally linear in W^b ($\Leftrightarrow$ algebraic)
- $\nabla_W V^a$ is linear in V^a
- $\nabla_W f = W(f)$
- $\nabla_W T$ is linear and Leibniz on tensor fields
- $\nabla_W \delta^b_a = 0$

Example: Coordinate Derivative

$$\partial_W V^a = \partial_W (V^\alpha b^a_\alpha) := W(V^\alpha)\, b^a_\alpha$$

∂^a_α (notation)

Transport: $\partial_W V^a = 0 \Leftrightarrow$ Keep the components constant!

The Space of Derivative Operators

Notation: $\nabla_W V^a =: W^b \nabla_b V^a$

emphasizes functional linearity in W^b

Let $\tilde{\nabla}_a$ and ∇_a denote two derivative operators:

$$(\tilde{\nabla}_a - \nabla_a)(f\omega_b) = \tilde{\nabla}_a(f\omega_b) - \nabla_a(f\omega_b)$$

$$= \omega_b \tilde{\nabla}_a f + f \tilde{\nabla}_a \omega_b$$

$$- \omega_b \nabla_a f - f \nabla_a \omega_b$$

$$= \omega_b [(df)_a - (df)_a] + f(\tilde{\nabla}_a - \nabla_a)\omega_b$$

$\Rightarrow (\tilde{\nabla}_a - \nabla_a)\omega_b$ is functionally linear in ω_b (algebraic)

$\Rightarrow (\tilde{\nabla}_a - \nabla_a)\omega_b = C_{ab}{}^c \omega_c$

tensor!

Example: Christoffel Symbols

∇_a = connection

∂_a = coordinate connection

$$(\nabla_a - \partial_a) w_b = \Gamma_{ab}{}^c w_c$$

Christoffel <u>tensor</u>

$\tilde{\partial}_a$ = <u>another</u> coordinate connection

$$(\nabla_a - \tilde{\partial}_a) w_b = \tilde{\Gamma}_{ab}{}^c w_c$$

<u>another</u> Christoffel <u>tensor</u>

$$\tilde{\Gamma}_{ab}{}^c w_c = \Gamma_{ab}{}^c w_c + (\partial_a - \tilde{\partial}_a) w_b$$

"non-tensorial" term

The Christoffel symbols $\Gamma_{\alpha\beta}{}^\gamma$ do not transform like a tensor because they are <u>different</u> tensors!

The set of all derivative operators on a manifold M is naturally an <u>affine space</u>:

Given ∇_a and $\tilde{\nabla}_a$, define

$$[\alpha \nabla_a + (1-\alpha)\tilde{\nabla}_a] T\ldots{}^{\ldots}$$

$$:= \alpha \nabla_a T\ldots{}^{\ldots} + (1-\alpha)\tilde{\nabla}_a T\ldots{}^{\ldots}$$

- $[\alpha \nabla_a + (1-\alpha)\tilde{\nabla}_a] f$

$$= \alpha \underbrace{\nabla_a f}_{(df)_a} + (1-\alpha)\underbrace{\tilde{\nabla}_a f}_{(df)_a} = (df)_a$$

- $[\alpha \nabla_a + (1-\alpha)\tilde{\nabla}_a]\delta_b^c = 0$, etc.

We can draw <u>straight lines</u> in the space of derivative operators, but there is no <u>natural</u> origin $\mathring{\nabla}_a$.

The connection is a <u>physical field</u>.

How do the actions of two derivative operators differ on other tensor fields?

Use the Leibniz property:

$$(\tilde{\nabla}_a - \nabla_a)\, V^b \cdot \omega_b$$

$$= \omega_b \tilde{\nabla}_a V^b - \omega_b \nabla_a V^b$$

$$= \tilde{\nabla}_a (\omega_b V^b) - V^b \tilde{\nabla}_a \omega_b$$

$$d(\omega_b V^b) \longrightarrow - \nabla_a (\omega_b V^b) + V^b \nabla_a \omega_b$$

$$= - V^b (\tilde{\nabla}_a - \nabla_a)\, \omega_b$$

$$= - V^b C_{ab}{}^{c}\, \omega_c$$

$$= - V^c C_{ac}{}^{b}\, \omega_b = - C_{ac}{}^{b} V^c \cdot \omega_b$$

$$(\tilde{\nabla}_a - \nabla_a)\, V^b = - C_{ac}{}^{b} V^c$$

$$\rightsquigarrow (\tilde{\nabla}_a - \nabla_a)\, T_{b_1 \cdots b_m}{}^{c_1 \cdots c_n}$$

$$= \sum_{i=1}^{m} C_{a b_i}{}^{d}\, T_{b_1 \cdots d \cdots b_m}{}^{c_1 \cdots c_n}$$

$$- \sum_{j=1}^{n} C_{ae}{}^{c_j}\, T_{b_1 \cdots b_m}{}^{c_1 \cdots e \cdots c_n}$$

Torsion

Let ∇_a be a derivative operator, and define the bracket

$$[V, W]_\nabla := \nabla_V W - \nabla_W V$$

of vector fields.

This bracket is not functionally linear in either argument:

$$[V, fW]_\nabla = \nabla_V (fW) - \nabla_{fW} V$$

$$= \nabla_V f \cdot W + f \nabla_V W - f \nabla_W V$$

$$= V(f) W + f [V, W]_\nabla$$

The ordinary Lie bracket has the <u>same</u> behavior

$$[V, fW](g) = V(fW(g)) - fW(V(g))$$

$$= V(f) \cdot W(g) + f V(W(g)) - f W(V(g))$$

$$\rightsquigarrow [V, fW] = V(f) W + f [V, W]$$

Neither $[v,w]_\nabla$ nor $[v,w]$ depends algebraically on the values of v and w at a point, but their difference does

$$[v,w]_\nabla - [v,w] = T(v,w)$$

linear map taking two vectors to one $\rightsquigarrow$ $\binom{1}{2}$ tensor field.

$\rightsquigarrow$ Torsion tensor $T_{ab}{}^c$

The torsion tensor is necessarily anti-symmetric $T_{(ab)}{}^c = 0$ because both brackets are.

$$T^{ab}{}_{c} = X^a Y^b w_c$$

$$\nabla_w (X^a Y^b w_c)$$

$$= Y^b w_c \nabla_w X^a$$

$$+ X^a w_c \nabla_w Y^b$$

$$+ X^a Y^b \nabla_w w_c$$

$$\tilde{T}(v,w) - T(v,w)$$

$$= [v,w]_{\tilde{\nabla}} - \cancel{[v,w]}$$

$$- [v,w]_{\nabla} + \cancel{[v,w]}$$

$$= \tilde{\nabla}_v w - \tilde{\nabla}_w v - \nabla_v w + \nabla_w v$$

$$\tilde{T}_{ab}{}^c - T_{ab}{}^c = -C_{ab}{}^c + C_{ba}{}^c$$

$$= -2 C_{[ab]}{}^c$$

$$v^a(\tilde{\nabla}_a - \nabla_a) w^c - w^a(\tilde{\nabla}_a - \nabla_a) v^c$$

$$= -v^a C_{ab}{}^c w^b + w^a C_{ab}{}^c v^b$$

$$= -v^a w^b (C_{ab}{}^c - C_{ba}{}^c)$$

The torsion tensor is also related to the commutator of covariant derivatives of functions.

$$V^a W^b T_{ab}{}^c \nabla_c f$$

$$= (\nabla_V W - \nabla_W V - [V,W])^c \nabla_c f$$

$$= (\nabla_V W^c - \nabla_W V^c - [V,W]^c) \nabla_c f$$

$$= \underbrace{\nabla_V (W^c \nabla_c f)}_{V(W(f))} - W^c \nabla_V \nabla_c f$$

$$- \underbrace{\nabla_W (V^c \nabla_c f)}_{W(V(f))} + V^c \nabla_W \nabla_c f$$

$$- [V,W](f)$$

$$= V^c W^d \nabla_d \nabla_c f - W^c V^d \nabla_d \nabla_c f$$

$$= -2 V^c W^d \nabla_{[c} \nabla_{d]} f$$

$$\boxed{2 \nabla_{[a} \nabla_{b]} f = - T_{ab}{}^c \nabla_c f}$$