# Lecture 9

## Derivative Operators

How do we do calculus with <u>tensor</u> fields?

# Tensor Algebra

1) addition, scalar multiplication

$$(\alpha T + \alpha' T')(v^1, \cdots, v^m, w_1, \cdots, w_n)$$
$$:= \alpha T(v^1, \cdots, v^m, w_1, \cdots, w_n)$$
$$+ \alpha' T'(v^1, \cdots, v^m, w_1, \cdots, w_n)$$

2) contraction

$$^{(11)}T(v^2, \cdots, v^m, w_2, \cdots, w_n)$$
$$:= \sum_\alpha T(b_\alpha, v^2, \cdots, v^m, \eta^\alpha, w_2, \cdots, w_n)$$

dual basis $\eta^\alpha(b_\beta) = \delta^\alpha_\beta$

contraction is basis-indep.

3) tensor product

$$T \otimes T'(v^1, \cdots v^{m+m'}, w_1, \cdots, w_{n+n'})$$
$$:= T(v^1, \cdots, v^m, w_1, \cdots w_n)$$
$$\times T'(v^{m+1}, \cdots, v^{m+m'}, w_{n+1}, \cdots, w_{n+n'})$$

# Abstract Index Notation

A $\binom{1}{1}$ tensor $T(V, \omega) = \#$ has three <u>adjoint actions</u>:

1) Maps dual vectors $\omega$ to dual vectors $T(\omega)$:

$$T(\omega)(V) := T(V, \omega) \quad \forall V$$

2) Maps vectors $V$ (dual dual vectors) to vectors $T(V)$:

$$T(V)(\omega) := T(V, \omega) \quad \forall \omega$$

3) Maps scalars $\alpha$ to $\binom{1}{1}$ tensors $T(\alpha)$:

$$T(\alpha)(V, \omega) := \alpha T(V, \omega)$$

More complicated tensors have many more adjoint actions.

The *abstract index notation*

Keeps track of all tensor actions using indices:

| | |
|---|---|
| scalar | $\alpha$ |
| vector | $V^a$ |
| dual vector | $\omega_a$ |
| $\binom{1}{1}$ tensor | $T_a{}^b$ |
| $\binom{m}{n}$ tensor | $T_{a_1 \cdots a_m}{}^{b_1 \cdots b_n}$ |

| | |
|---|---|
| scalar multiplication | $[\alpha V]^a = \alpha V^a$ |
| natural pairing | $\omega(V) = \omega_a V^a$ |
| tensor product | $[\omega \otimes V]_a{}^b = \omega_a V^b$ |
| tensor action | $T(V, \omega) = T_a{}^b V^a \omega_b$ |
| → vector map | $[T(V)]^a = T_b{}^a V^b$ |
| → dual vector map | $[T(\omega)]_a = T_a{}^b \omega_b$ |
| → scalar map | $[\alpha T]_a{}^b = \alpha T_a{}^b$ |

contraction $\quad {}^{(11)}T = T_a{}^a$

Notes:

- Different tensors are denoted with different stem letters

$$V^a, \; W^a, \; V^b, \; W^c, \; \ldots$$

- Different indices label representatives of a given vector in different <u>copies</u> of the vector space:

$$V^a, \; V^b, \; V^c, \; \text{etc.}$$

Note: $V^a + W^b$ <u>makes no sense</u>!

- All copies of the space are isomorphic under the identity map $\delta^a_b$:

$$W^a = \delta^a_b \, W^b$$

contraction

$$\rightsquigarrow \delta^a_b T_a{}^b = T_a{}^a = {}^{(11)}T$$

- The order of stem letters doesn't matter, only how they pair with indices:

$$V^a w_b = w_b V^a \neq V_b w^a$$

makes no sense!

$V =$ vector
$W =$ co-vec.

- Vector and co-vector indices generally commute, but vector and vector indices do not:

$$T^{ab}{}_c = T^a{}_c{}^b = T_c{}^{ab}$$

$$\neq T^{ba}{}_c \leftarrow T(\omega_1, \omega_2) \neq T(\omega_2, \omega_1)$$

- Symmetric and anti-symmetric parts are denoted with brackets:

$$T^{(a_1 \cdots a_n)} = \frac{1}{n!} \sum_\pi T^{a_{\pi(1)} \cdots a_{\pi(n)}}$$

$$T^{[a_1 \cdots a_n]} = \underbrace{\frac{1}{n!} \sum_\pi}_{\text{average over permutations}} \underbrace{(-1)^\pi}_{\text{even/odd permutation.}} T^{a_{\pi(1)} \cdots a_{\pi(n)}}$$

# Concrete Indices

A basis is a collection of vectors $\{b_\alpha\} \rightsquigarrow b_\alpha^{\ a}$ ← abstract, ← concrete

with $\alpha = 1, \ldots, n$ such that every vector $V^a$ can be written uniquely in the form

$$V^a = \sum_\alpha V^\alpha\, b_\alpha^{\ a}$$

↖ scalar components

We denote the dual basis $b^\alpha_{\ a}$ in abstract index notation:

$$b^\alpha_{\ a} V^a := V^\alpha \quad \Longleftrightarrow \quad \boxed{b^\alpha_{\ a}\, b_\beta^{\ a} = \delta^\alpha_{\ \beta}}$$

$$\Updownarrow$$

$$V^a = \sum_\alpha \left(b^\alpha_{\ b} V^b\right) b_\alpha^{\ a}$$

for all $V$ → $$\delta^a_{\ b} V^b = \sum_\alpha \left(b_\alpha^{\ a}\, b^\alpha_{\ b}\right) V^b$$

$$\Downarrow$$

contraction calculation → $$\boxed{\sum_\alpha b_\alpha^{\ a}\, b^\alpha_{\ b} = \delta^a_{\ b}}$$

- <u>Tensor components</u> are defined by contracting with appropriate basis and dual basis elements:

$$T^{ab}{}_{cd}\, b^{\alpha}_{a}\, b^{\beta}_{b}\, b^{c}_{\gamma}\, b^{d}_{\delta} = T^{\alpha\beta}{}_{\gamma\delta}$$

$$T^{\alpha\beta}{}_{\gamma\delta}\, b^{a}_{\alpha}\, b^{b}_{\beta}\, b^{\gamma}_{c}\, b^{\delta}_{d} = T^{ab}{}_{cd}$$

↖ sum implicit (Einstein conv.)

- Contractions can be calculated by taking traces of the corresponding arrays of numerical components:

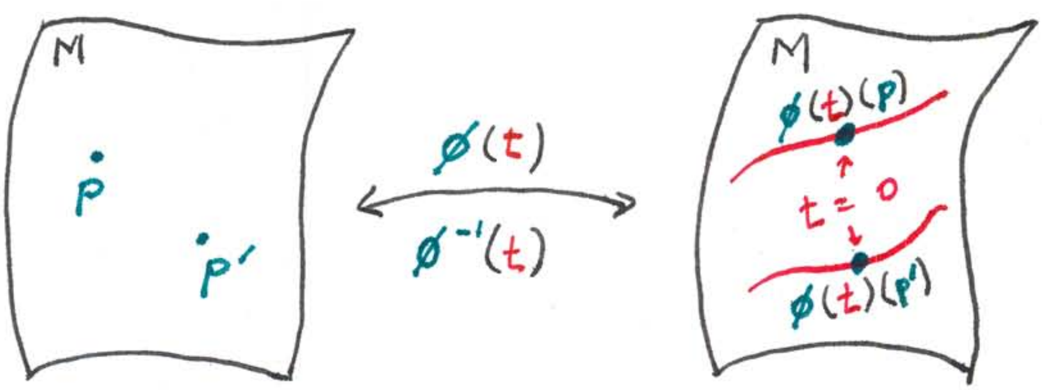
$$w_a V^a = w_a \delta^a_b V^b$$
$$= w_a b^a_\alpha b^\alpha_b V^b = w_\alpha V^\alpha$$

⤳ It is trivial to convert abstract indices to concrete and back. Abstract indices are useful because the are <u>manifestly</u> coordinate-indep.

## Lie Derivative

Let $\phi(t)$ denote a smooth <u>one-parameter group</u> of <u>diffeomorphisms</u> from a manifold $M$ to itself:

- $\phi(t): M \to M$ is smooth with smooth inverse
- for each fixed $p \in M$, and variable $t$, $\phi(t)(p)$ is a smooth curve in $M$
- $\phi(0) = \text{identity}: M \to M$
- $\phi(t) \circ \phi(t') = \phi(t+t')$




$$\underbrace{\dot{\phi}(p)}(f) := \frac{d}{dt} f(\phi(t)(p))\Big|_{t=0}$$

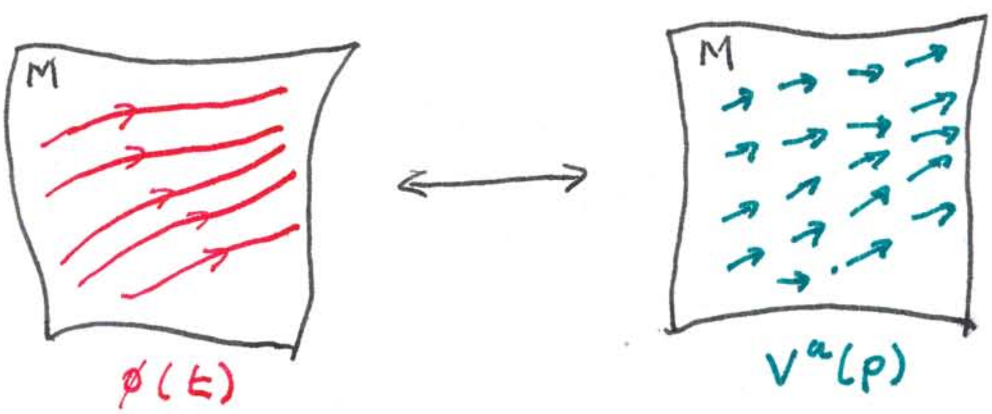
tangent vector at $p$.

The vector field $\dot{\phi}^a$ is tangent to the <u>flow</u> through M defined by $\phi(t)$.

Conversely, given a smooth vector field $V^a$, we can solve the ODE

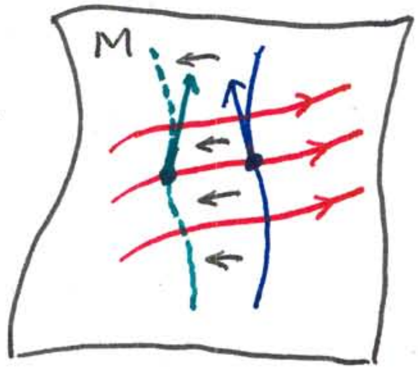
$$\dot{\gamma}^a(t) = V^a(\gamma(t))$$

starting from each $p \in M$ to get a smooth flow of <u>integral curves.</u>




Vector fields are <u>infinitesimal generators</u> of diffeomorphisms!

The diffeomorphism $\phi(t)$ maps tensor fields $T$ to tensor fields $\phi(t) \cdot T$.




Idea: "Let an object there act on objects here. Move it using $\phi(t)$."

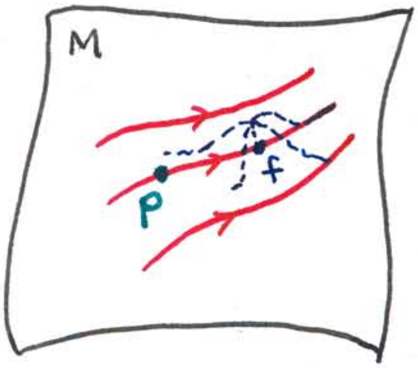
## Scalar Fields

$$[\phi(t) \cdot f](\text{here}) = f(\text{there})$$

$$[\phi(t) \cdot f](p) := f(\phi(t)(p))$$

$$\rightsquigarrow \frac{d}{dt}[\phi(t) \cdot f](p) = \frac{d}{dt} f(\phi(t)(p))$$

$$=: \dot{\phi}(f)(p)$$



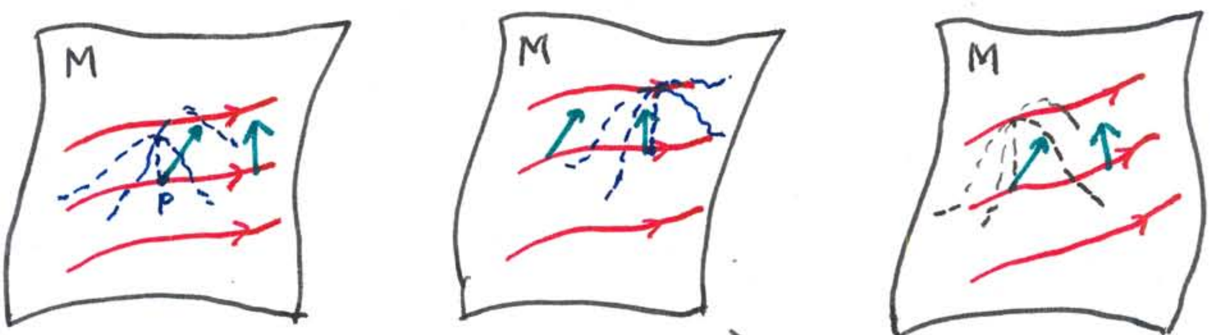
## Vector Fields

$[\phi(t)\cdot V](p)$ must act on the function $f$ here using the value of the vector field there.

⇝ slide the function forward

$$\underbrace{[\phi(t)\cdot V](p)}_{\text{vector at } p}(f) := \underbrace{V(\phi(t)(p))}_{\substack{\text{vector at}\\ \phi(t)(p)}}\,(\underbrace{\phi(-t)\cdot f}_{\substack{f \text{ slid}\\ \text{forward}}})$$

$$\leadsto \underbrace{[\phi(t)\cdot V](f)}_{\text{smooth fcn.}} = \underset{\substack{\uparrow\\ \text{slide}\\ \text{back}}}{\phi(t)}\cdot\left[\underbrace{V(\phi(-t)\cdot f)}_{\text{smooth fcn.}}\right]$$




slide the function forward → act! → slide the result back

$$\frac{d}{dt}[\phi(t)\cdot V](f) = \dot\phi(V(f)) + V(-\dot\phi(f))$$
$$= [\dot\phi, V](f)$$

## Co-Vector Fields

$$\underbrace{[\phi(t)\cdot\omega](p)}_{\text{co-vector at } p}\underset{\substack{\uparrow \\ \text{at } p}}{(V)} := \underbrace{\omega(\phi(t)(p))}_{\text{co-vector at } \phi(t)(p)}\underbrace{(\phi(-t)\cdot V)}_{\text{slid forward to } \phi(t)(p)}$$


If we now let $V$ be a vector <u>field</u>, then

$$\underbrace{[\phi(t)\cdot\omega](V)}_{\text{function.}} = \underset{\substack{\nearrow \\ \text{slide back}}}{\phi(t)}\cdot\underset{\substack{\uparrow \\ \text{act}}}{[\omega}(\underset{\substack{\nwarrow \\ \text{slide forward}}}{\phi(-t)\cdot V})]$$

$$\frac{d}{dt}[\phi(t)\cdot\omega](V) = \dot\phi(\omega(V)) + \omega(-[\dot\phi, V])$$
$$= \dot\phi(\omega(V)) - \omega([\dot\phi, V])$$

action on function $\omega(V)$ <u>minus</u>
$\omega$(action on vector field $V$)

## Axiomatic Definition of $\mathcal{L}_{\dot{\phi}}$

- $\mathcal{L}_{\dot{\phi}} f := \dot{\phi}(f)$
- $\mathcal{L}_{\dot{\phi}} V^a := [\dot{\phi}, V]^a$
- $(\mathcal{L}_{\dot{\phi}} \omega_a) V^a = \mathcal{L}_{\dot{\phi}} (\omega_a V^a) - \omega_a \mathcal{L}_{\dot{\phi}} V^a$

Leibniz property

$$\mathcal{L}_{\dot{\phi}} (T \otimes T') = \mathcal{L}_{\dot{\phi}} T \otimes T' + T \otimes \mathcal{L}_{\dot{\phi}} T'$$

$\rightsquigarrow$ Define $\mathcal{L}_{\dot{\phi}}$ for scalars and vector fields, and extend to all tensors by

(a) linear

(b) Leibniz.