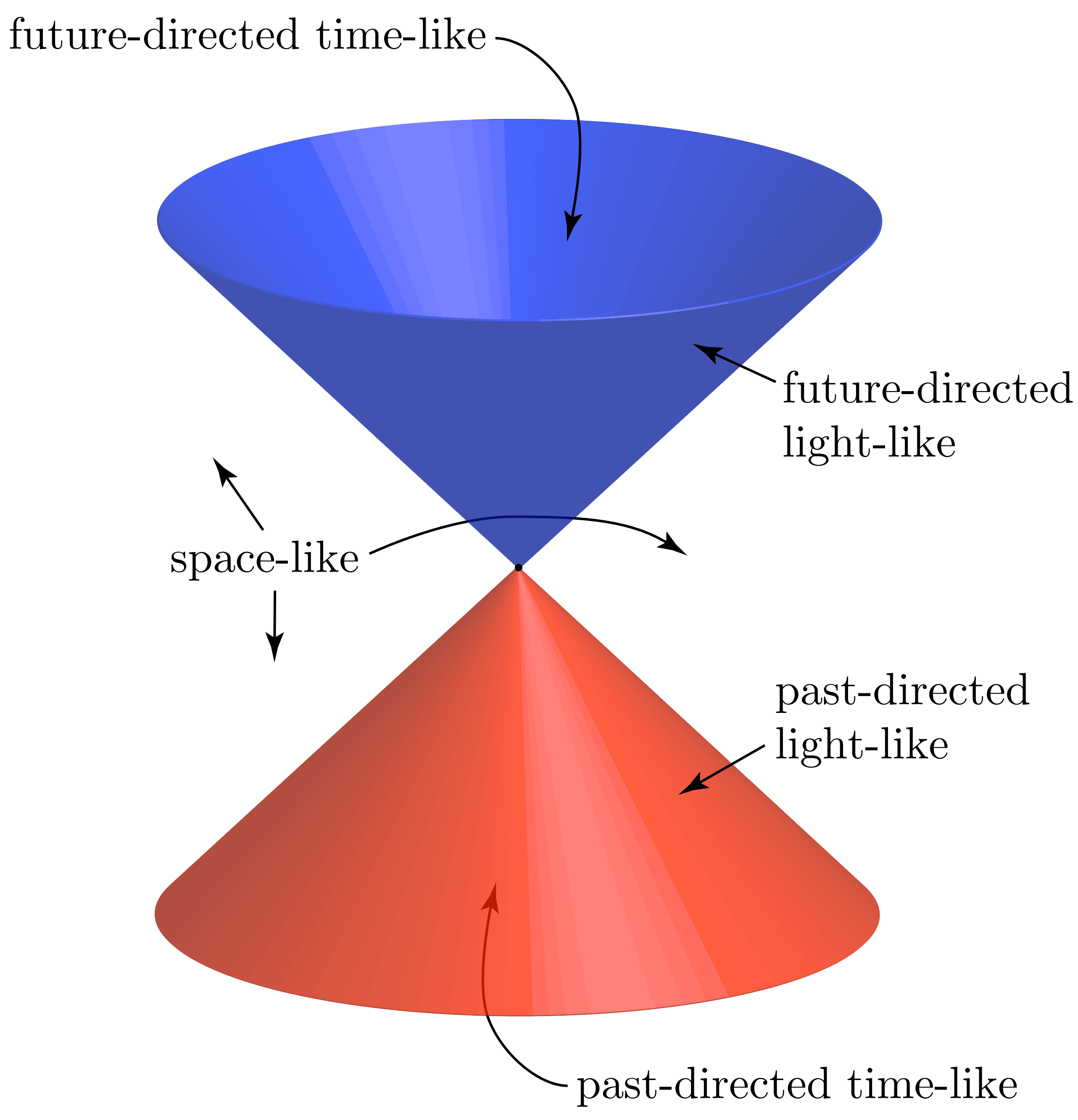
Relativistic Kinematics

Lecture II
General Relativity (PHY 6938), Fall 2007

FAU
Florida Atlantic University

Structure of Minkowski Space

- There are five kinds of vectors in Minkowski spacetime:
  - space-like $\|\mathbf{x}\|^2 > 0$
  - time-like $\|\mathbf{x}\|^2 < 0$
  - and light-like. $\|\mathbf{x}\|^2 = 0$
  - Time-like and light-like vectors can be either
    - future-directed $t_{\mathbf{x}} > 0$
    - or past-directed. $t_{\mathbf{x}} < 0$



$$\|\mathbf{x}\|^2 := -c^2\, t_{\mathbf{x}}^2 + |\vec{x}|^2$$

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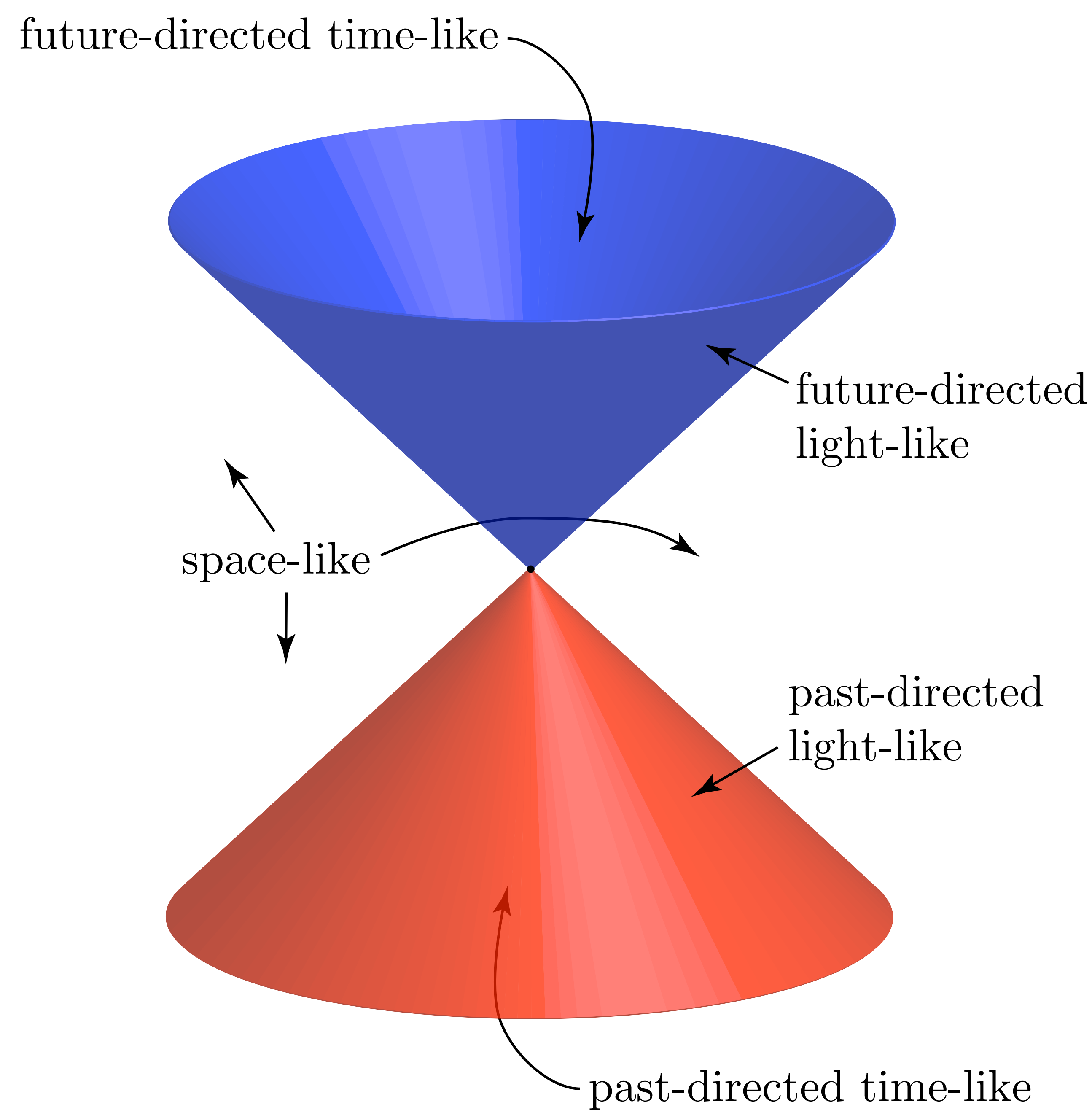
$$t' = \gamma\,(t - \vec{v}\cdot\vec{x}/c^2)$$

$$|\vec{v}\cdot\vec{x}| \leq |\vec{v}|\,|\vec{x}| < vc\,|t_{\mathbf{x}}| < c^2\,|t_{\mathbf{x}}|$$



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Causal

$$t' = \gamma\,(t - \vec{v}\cdot\vec{x}/c^2)$$

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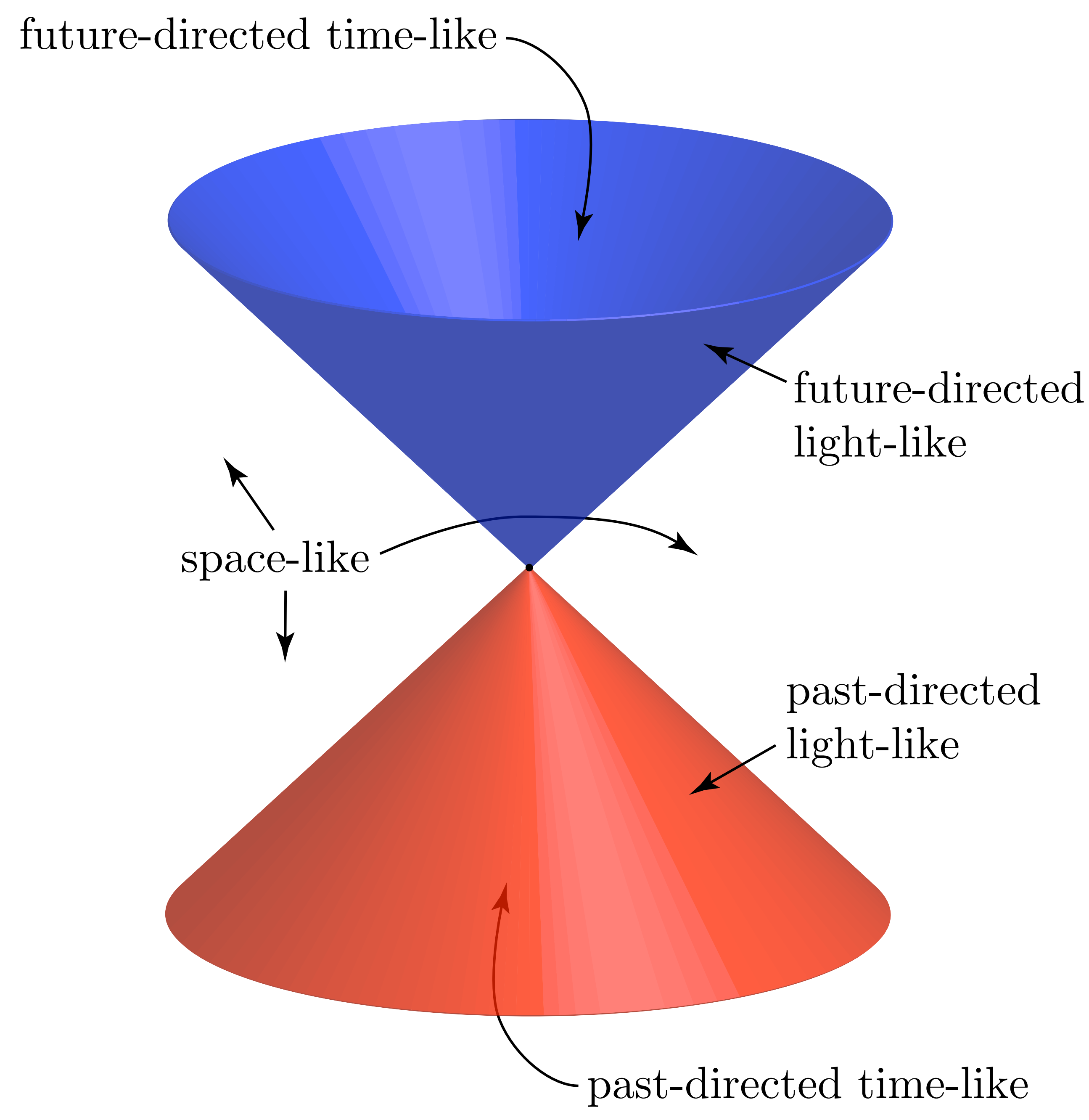

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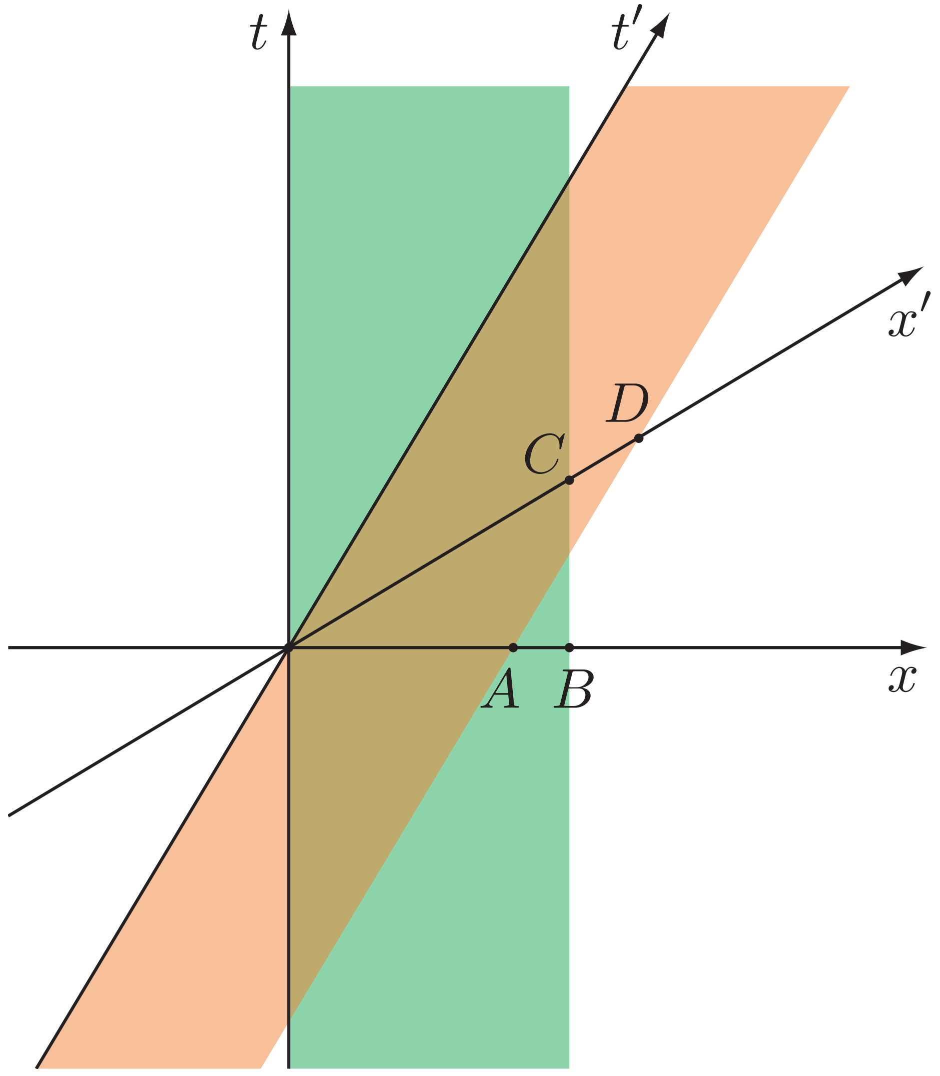
Kinematical Effects

Length Contraction

An inertial observer O' carries a ruler of length L_0 at speed v past an inertial observer O.

How long does O measure it to be?



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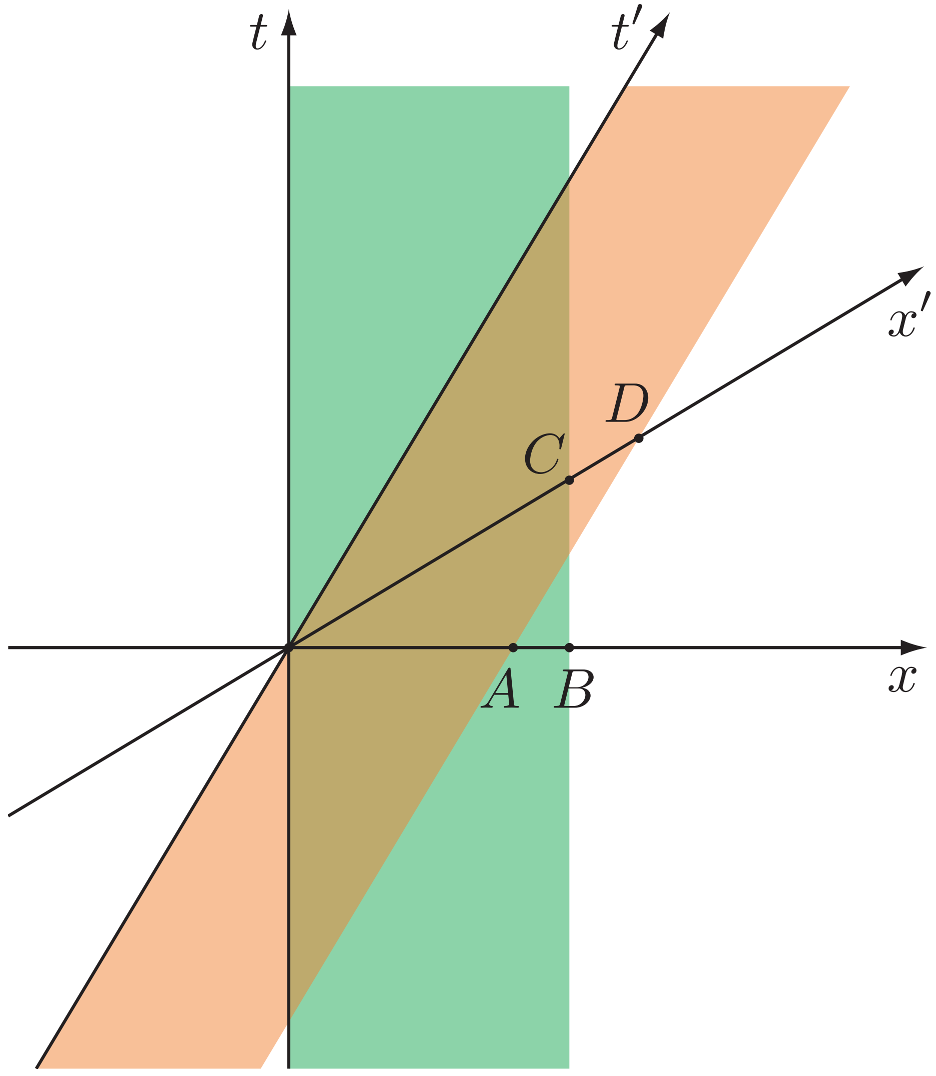
How long does O' measure and identical ruler carried by O to be?



$$x'_0 = 0 = \gamma\,(x_0 - vt)$$

$$x'_1 = L_0 = \gamma\,(x_1 - vt)$$

$$x_1(t) - x_0(t) = \frac{L_0}{\gamma} = \sqrt{1 - v^2/c^2}\,L_0$$

$$x_1(t) - x_0(t) = \frac{L_0}{\gamma} = \sqrt{1 - v^2/c^2}\, L_0$$

Length Contraction

An inertial observer *O′* carries a ruler of length L_0 at speed *v* past an inertial observer *O*.

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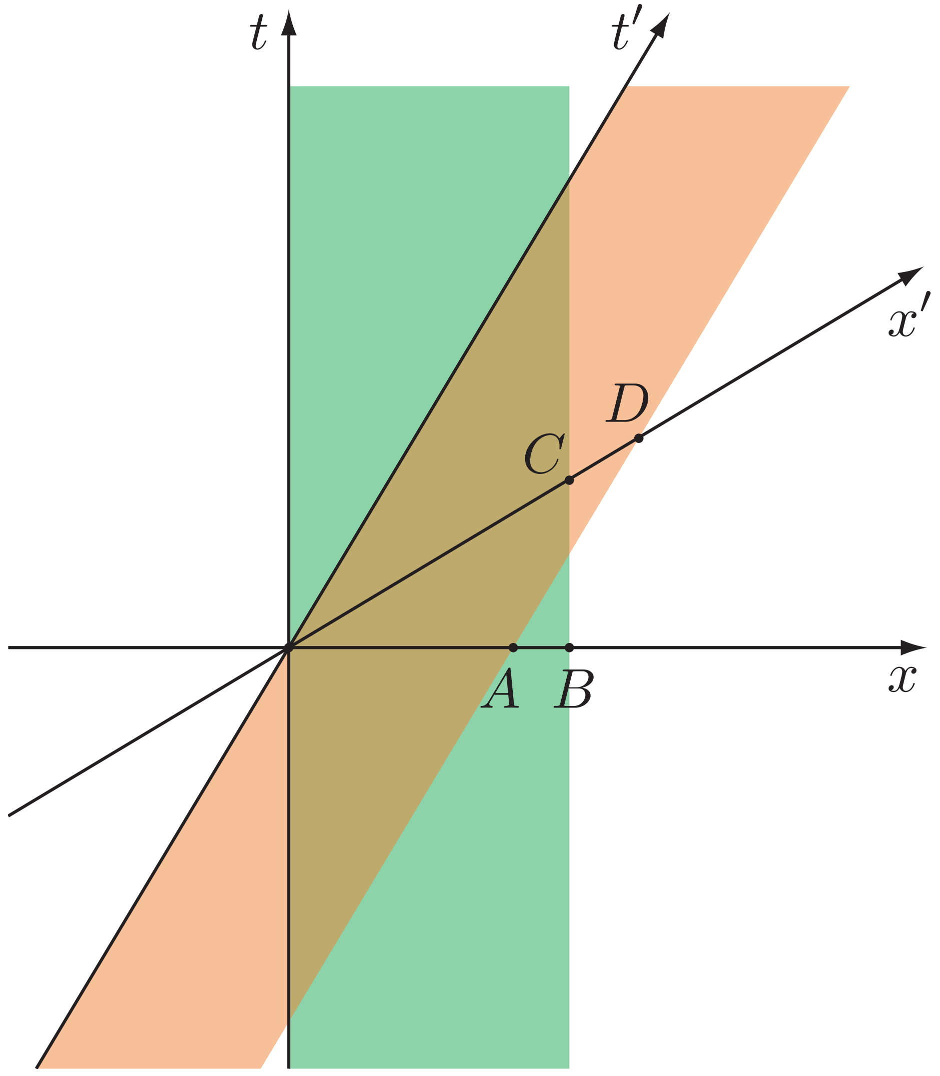

$$x_0 = 0 \qquad x' = \gamma\,(x - vt)$$

$$x_1 = L_0 \qquad t' = \gamma\,(t - vx/c^2)$$

$$x' + vt' = \gamma\,(x - v^2 x/c^2)$$

$$x_1(t) - x_0(t) = \frac{L_0}{\gamma} = \sqrt{1 - v^2/c^2}\, L_0$$

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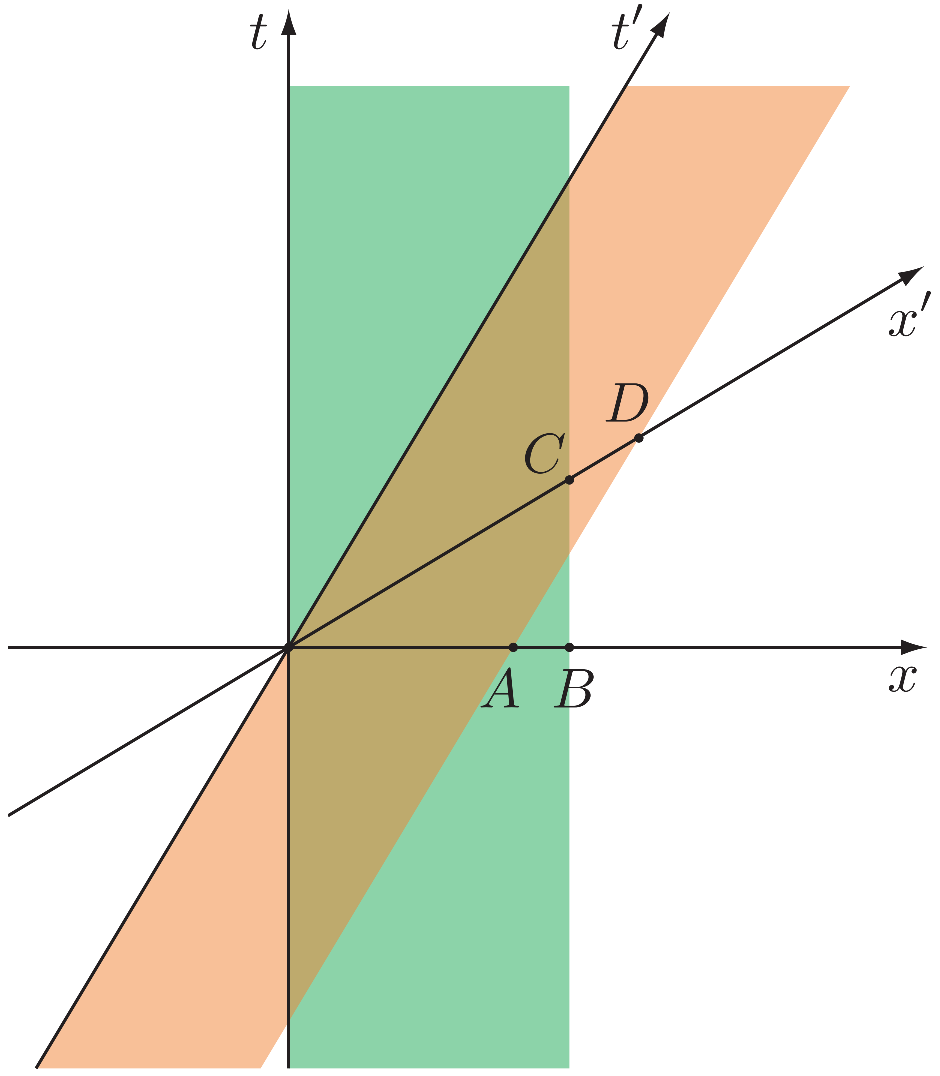

$$x_0 = 0$$

$$x_1 = L_0$$

$$x' + vt' = \gamma\left(x - v^2 x/c^2\right) = \frac{x}{\gamma}$$

$$x_1'(t') - x_0'(t') = \frac{L_0}{\gamma}$$

$$x_1(t) - x_0(t) = \frac{L_0}{\gamma} = \sqrt{1 - v^2/c^2}\, L_0$$

$$x'_1(t') - x'_0(t') = \frac{L_0}{\gamma}$$

Length Contraction

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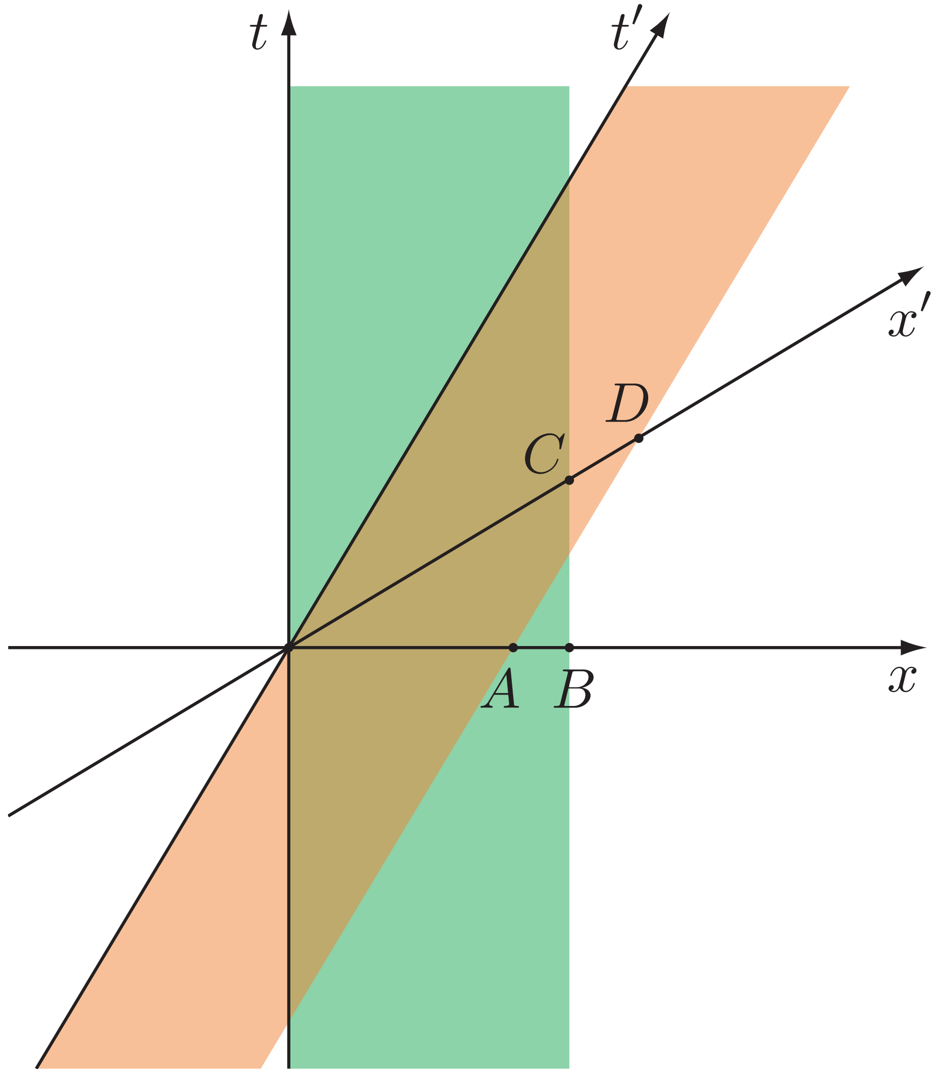
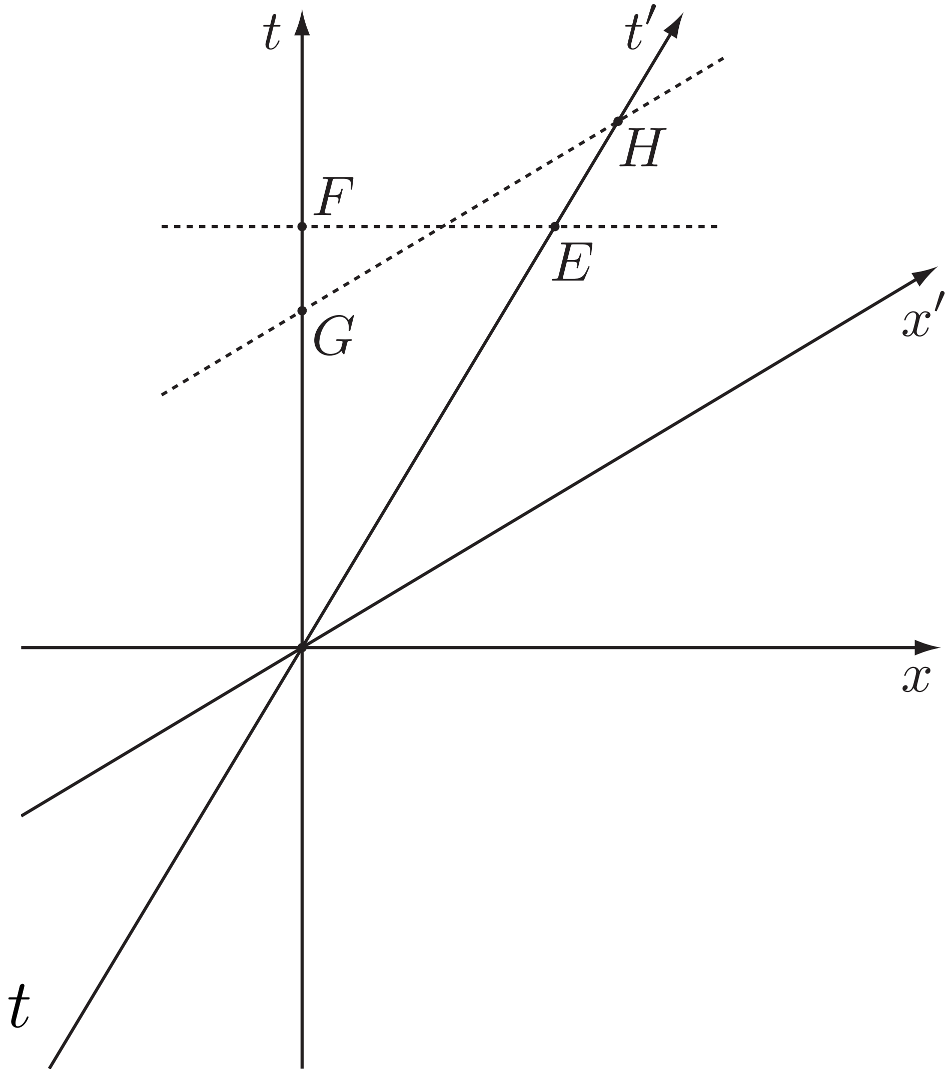


Length of O′ ruler measured by O = ||A|| < ||B||

Length of O ruler measured by O′ = ||C|| < ||D||

$$T := t_E = \gamma T_0 = \frac{T_0}{\sqrt{1 - v^2/c^2}}$$



$$\vec{x}_{O'}(t) = \vec{v}\,t$$

Time Dilation

An inertial observer *O′* carries a clock that advances a time T_0 while she passes *O* at speed *v*.

How much time elapses for *O*?

What happens if the roles are reversed?

$$t'_E = \gamma\,(t_E - \vec{v}\cdot\vec{x}_E/c^2) = \gamma\,(1 - v^2/c^2)\,t_E = \frac{t_E}{\gamma}$$

$$T := t_E = \gamma T_0 = \frac{T_0}{\sqrt{1 - v^2/c^2}}$$

$$T' = \gamma T_0$$

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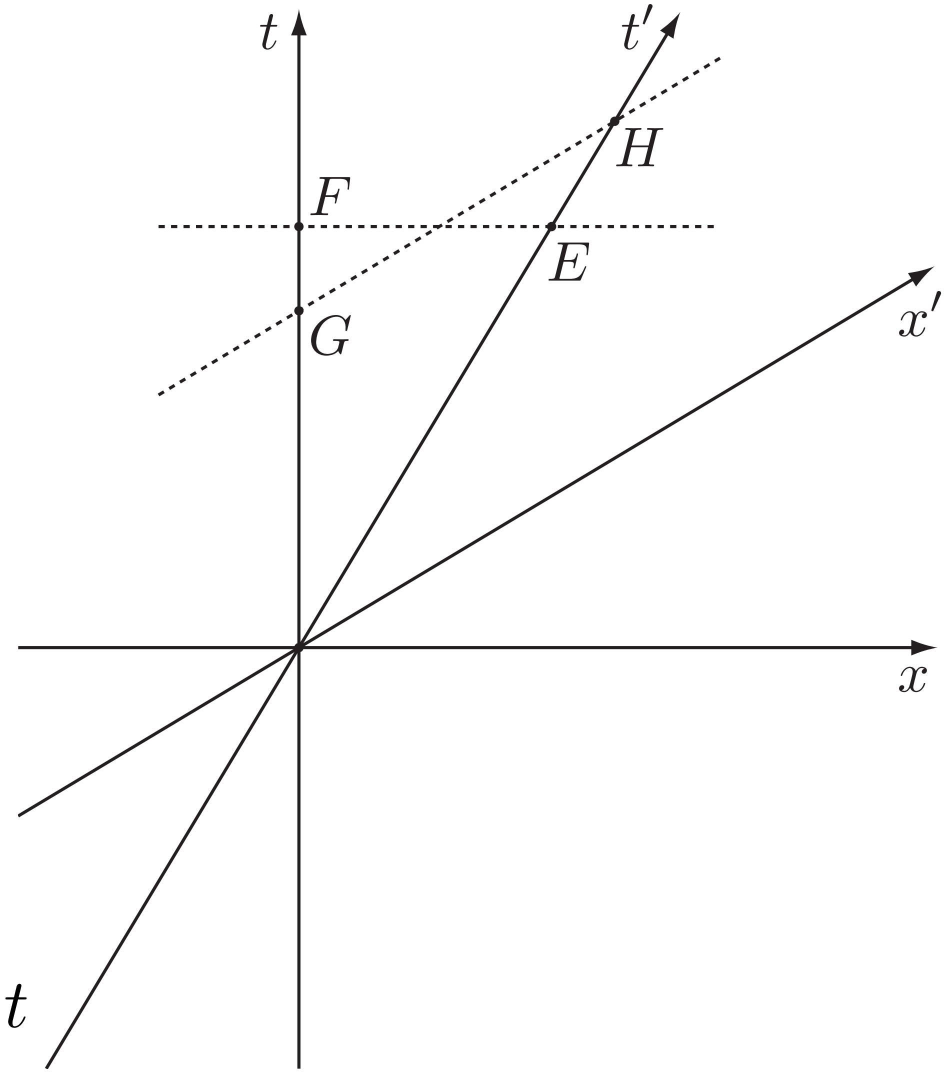

$$\vec{x}_{O'}(t) = \vec{v}\,t$$

$$t'_G = \gamma\,(t_G - \vec{v}\cdot\vec{x}_G/c^2) = \gamma T_0$$

$$T := t_E = \gamma T_0 = \frac{T_0}{\sqrt{1 - v^2/c^2}}$$

$$T' = \gamma T_0$$

Time Dilation

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How much time elapses for *O*?

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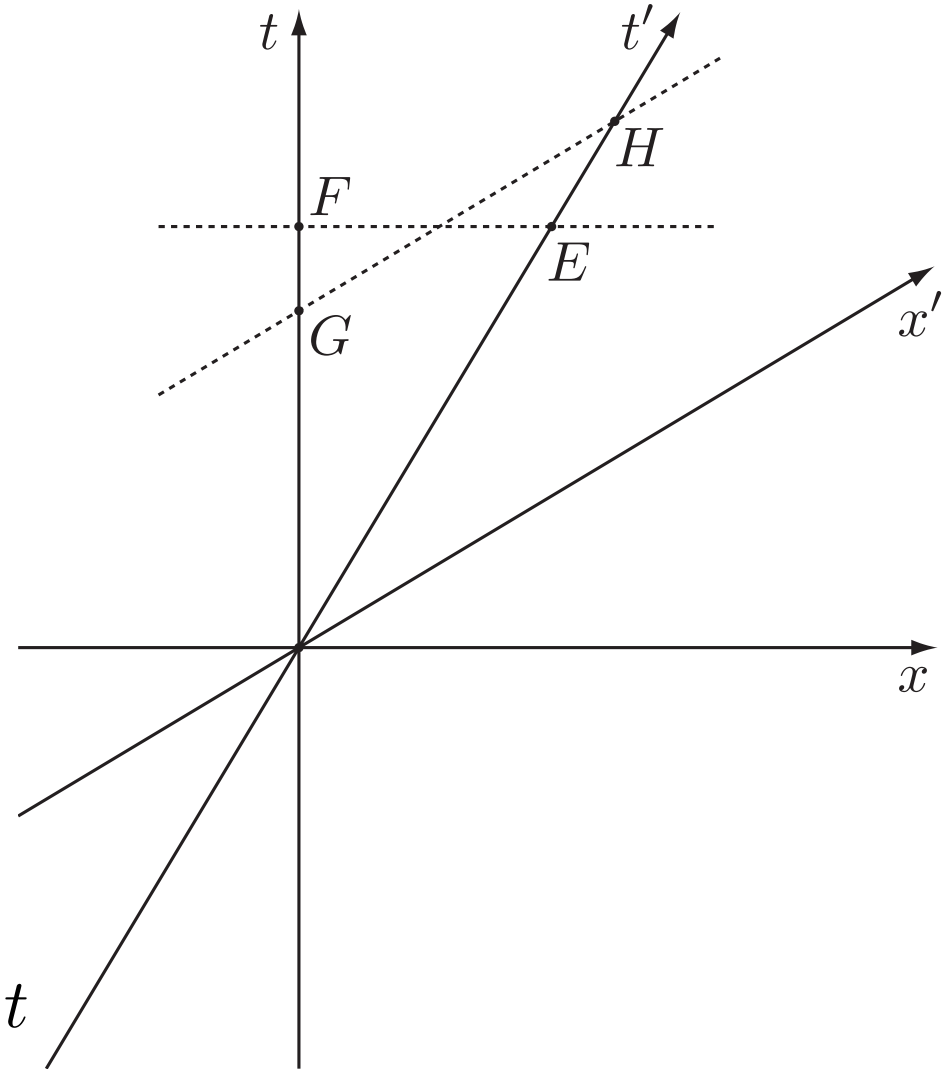


$$\vec{x}_{O'}(t) = \vec{v}\,t$$

$$\sqrt{-\|F\|^2} > \sqrt{-\|G\|^2} = \sqrt{-\|E\|^2} < \sqrt{-\|H\|^2}$$

$$\left[I + (\gamma - 1)\,\hat{v}\hat{v} + \vec{u}\,'\vec{v}/c^2\right]^{-1} = I - \frac{(1-\gamma^{-1})\,\hat{v}\hat{v}}{1+\vec{v}\cdot\vec{u}\,'/c^2} - \frac{\vec{u}\,'\vec{v}/c^2}{1+\vec{v}\cdot\vec{u}\,'/c^2}$$

Exercise!

Velocity Addition

A particle moves with uniform velocity *u*′ relative to *O*′, who moves with uniform velocity *v* relative to *O*.

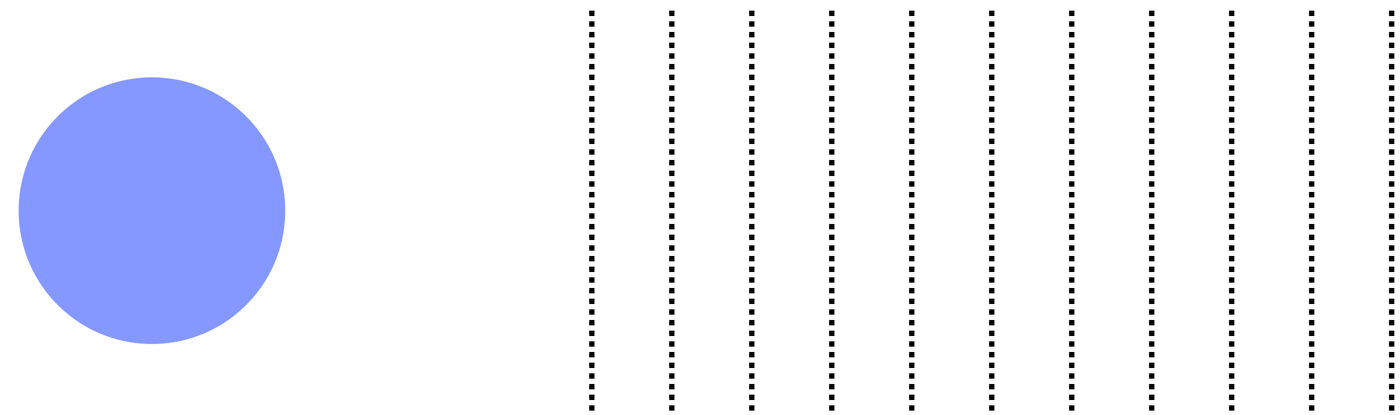
This particle will move with uniform velocity *u* relative to *O*. What is it?

$$\vec{x}\,' = \vec{u}\,'\,t'$$

$$\left[I + (\gamma - 1)\,\hat{v}\hat{v}\right]\cdot\vec{x} - \gamma\,\vec{v}\,t = \vec{u}\,'\,\gamma\left(t - \vec{v}\cdot\vec{x}/c^2\right)$$

$$\left[I + (\gamma - 1)\,\hat{v}\hat{v} + \vec{u}\,'\vec{v}/c^2\right]\cdot\vec{x} = \gamma\left(\vec{v} + \vec{u}\,'\right)t$$

$$\vec{x} = \vec{u}\,t \qquad \text{with} \qquad \vec{u} = \frac{\vec{v} + \vec{u}\,'_{\parallel} + \vec{u}\,'_{\perp}/\gamma}{1+\vec{v}\cdot\vec{u}\,'/c^2}$$

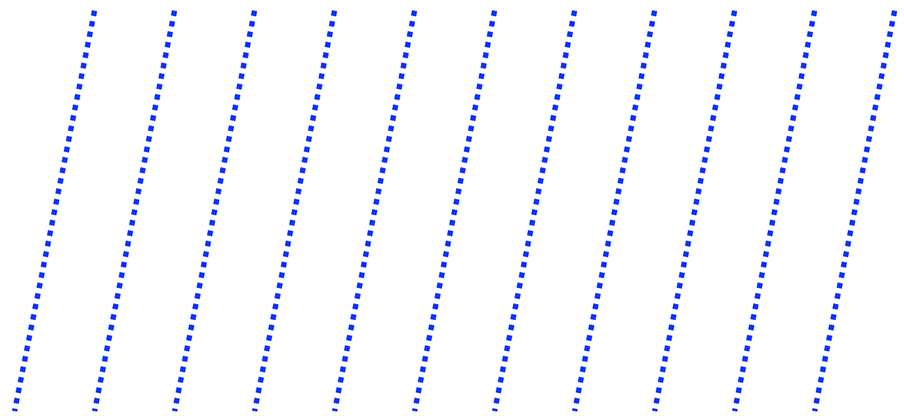


Aberration

How does the angle of incidence of a stream of particles depend on the motion of the observer?

How does the angle of incidence of a wave depend on the motion of the observer?

Aberration



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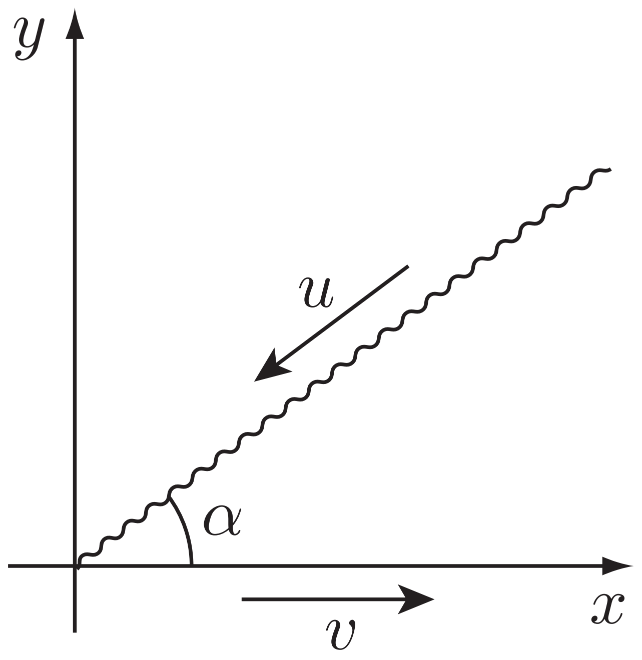

Aberration

How does the angle of incidence of a stream of particles depend on the motion of the observer?

How does the angle of incidence of a wave depend on the motion of the observer?

$$\frac{u'^2}{c^2} = 1 - \frac{(1 - v^2/c^2)\,(1 - u^2/c^2)}{[1 + (vu/c^2)\cos\alpha]^2}$$

$$\vec{u}\,' = \frac{\vec{u}_{\parallel} - \vec{v} + \vec{u}_{\perp}/\gamma}{1 - \vec{v}\cdot\vec{u}/c^2}$$

$$u'\cos\alpha' = \frac{u\cos\alpha + v}{1 + (vu/c^2)\cos\alpha}$$

$$u'\sin\alpha' = \frac{(u/\gamma)\sin\alpha}{1 + (vu/c^2)\cos\alpha}$$

$$\tan\alpha' = \frac{\sin\alpha}{\gamma\,(\cos\alpha + v/u)}$$

Aberration

How does the angle of incidence of a stream of particles depend on the motion of the observer?

How does the angle of incidence of a **wave** depend on the motion of the observer?

(Particle Case)

$$\tan\alpha' = \frac{\sin\alpha}{\gamma\,(\cos\alpha + v/u)}$$

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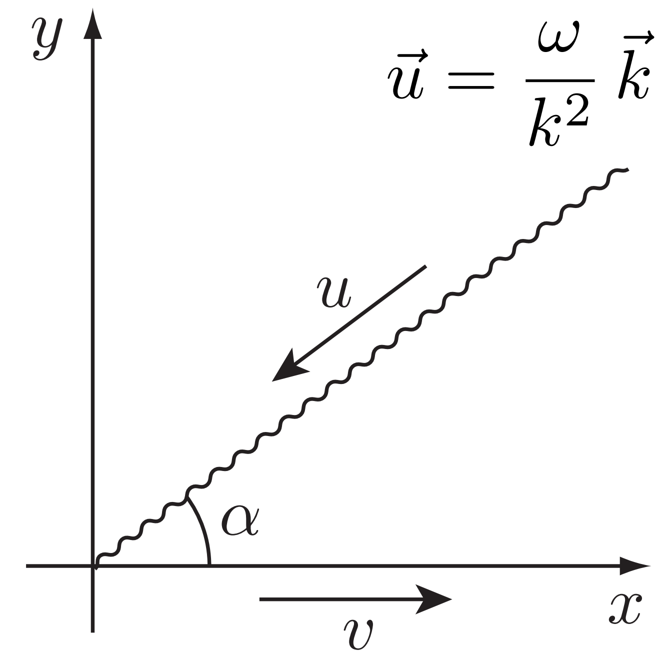

$$\vec{u} = \frac{\omega}{k^2}\,\vec{k}$$

$$\Psi(t, \vec{x}) = A\,\mathrm{e}^{-\mathrm{i}(\omega t - \vec{k}\cdot\vec{x})}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\,(\omega t - \vec{k}\cdot\vec{x}) = \omega - \vec{k}\cdot\vec{u}$$

$$\omega t - \vec{k}\cdot\vec{x} = \omega\,\gamma\left(t' + \vec{v}\cdot\vec{x}\,'/c^2\right) - \vec{k}\cdot\left[\vec{x}\,' + (\gamma - 1)\,\hat{v}\hat{v}\cdot\vec{x}\,' + \gamma\,\vec{v}\,t\right]$$

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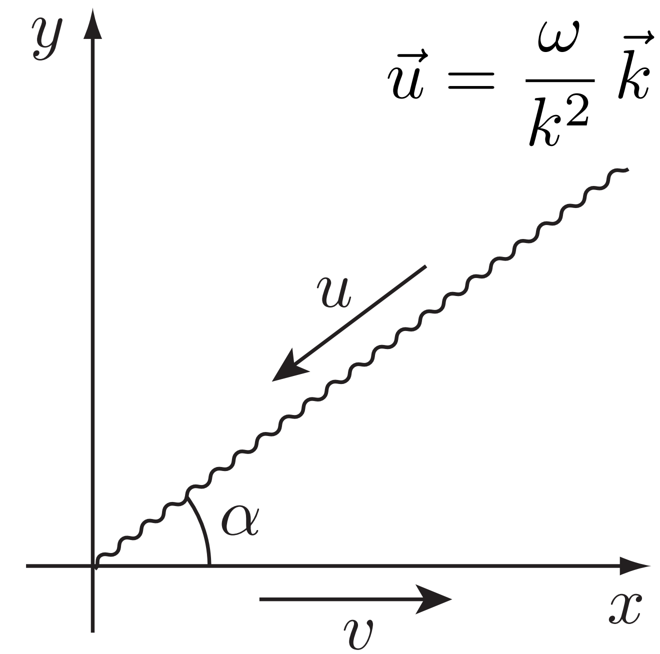

$$\vec{u} = \frac{\omega}{k^2}\,\vec{k}$$

Aberration

$$\Psi(t, \vec{x}) = A\,\mathrm{e}^{-\mathrm{i}(\omega t - \vec{k}\cdot\vec{x})}$$

How does the angle of incidence of a stream of particles depend on the motion of the observer?

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\,(\omega t - \vec{k}\cdot\vec{x}) = \omega - \vec{k}\cdot\vec{u}$$

How does the angle of incidence of a wave depend on the motion of the observer?

$$\begin{pmatrix} \omega'/c^2 \\ \vec{k}' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma c^{-2}\,\vec{v}\,\cdot \\ -\gamma\,\vec{v} & I\,\cdot\; + (\gamma - 1)\,\hat{v}\hat{v}\,\cdot \end{pmatrix} \begin{pmatrix} \omega/c^2 \\ \vec{k} \end{pmatrix}$$

$$\omega t - \vec{k}\cdot\vec{x} = \gamma\,\left(\omega - \vec{v}\cdot\vec{k}\right)t' - \left[\vec{k} + (\gamma - 1)\,\vec{k}\cdot\hat{v}\hat{v} - \gamma\,\vec{v}\,\omega/c^2\right]\cdot\vec{x}'$$

(Particle Case)

$$\tan\alpha' = \frac{\sin\alpha}{\gamma\left(\cos\alpha + v/u\right)}$$

$$\frac{u'^2}{c^2} = 1 - \frac{\left(1 - v^2/c^2\right)\left(1 - u^2/c^2\right)}{\left[1 + \left(vu/c^2\right)\cos\alpha\right]^2}$$

Aberration

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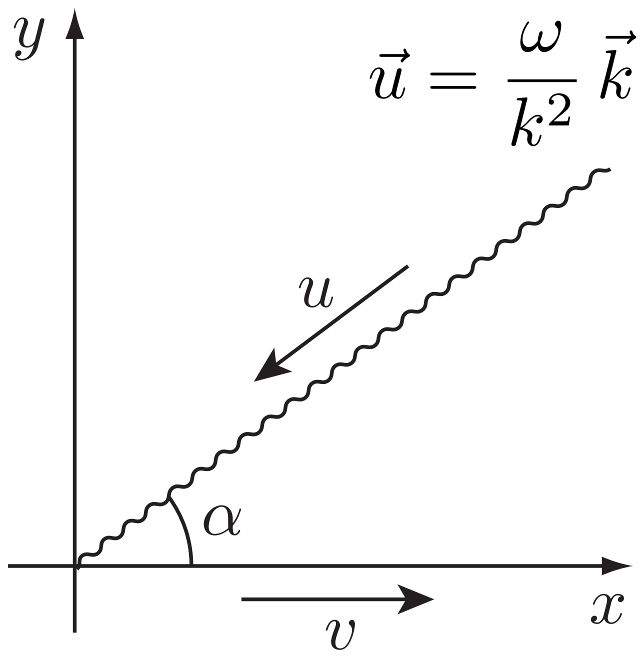

$$\omega' = \gamma\left(\omega + vk\cos\alpha\right)$$

$$k'\cos\alpha' = \gamma\left(k\cos\alpha + v\omega/c^2\right)$$

$$k'\sin\alpha' = k\sin\alpha$$

(Particle Case) $u \leftrightarrow \frac{c^2}{u}$ (Wave Case)

$$\tan\alpha' = \frac{\sin\alpha}{\gamma\left(\cos\alpha + v/u\right)}$$

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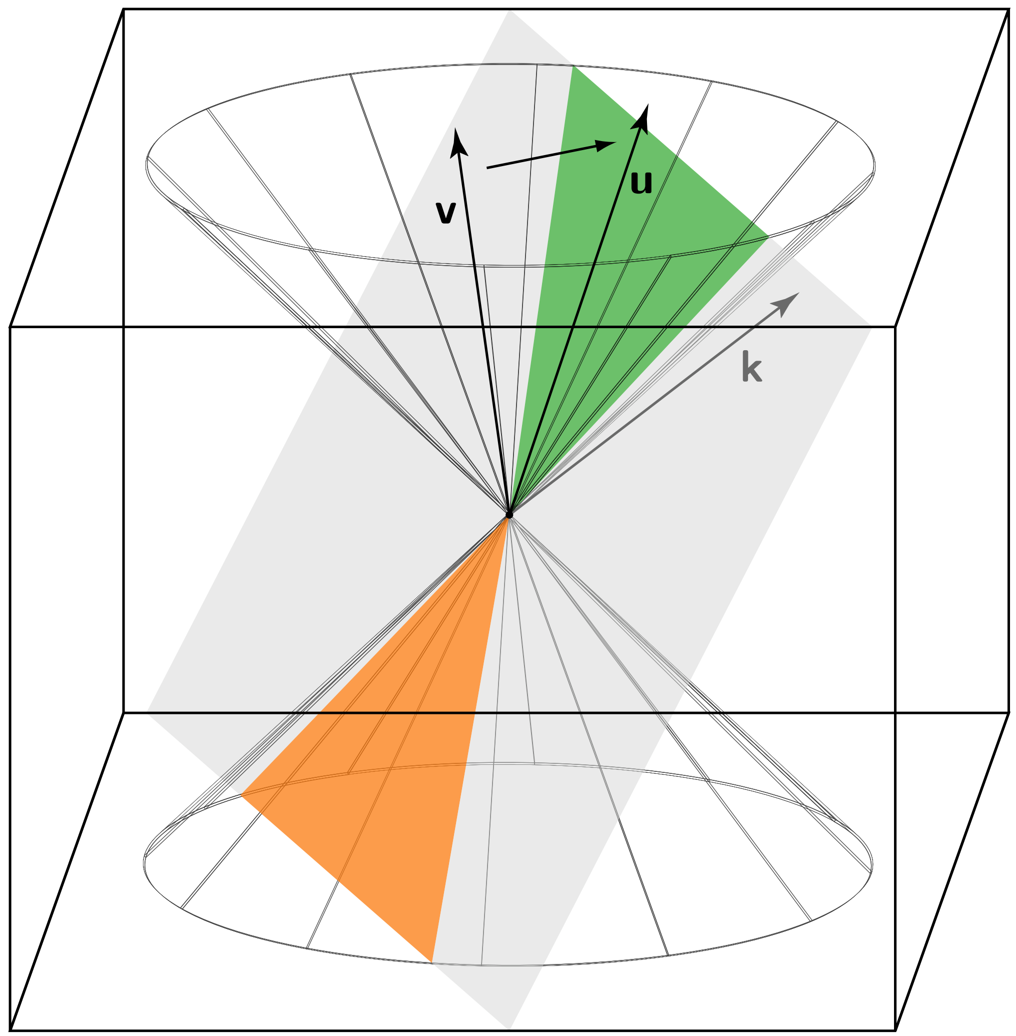
Aberration

How does the angle of incidence of a stream of particles depend on the motion of the observer?

How does the angle of incidence of a wave depend on the motion of the observer?

$$\mathbf{u} = \frac{\mathbf{v} - \hat{\mathbf{k}}\,\hat{\mathbf{k}} \cdot \mathbf{v}}{\sqrt{1 + \left(\hat{\mathbf{k}} \cdot \mathbf{v}/c\right)^2}}$$



Doppler Shift

Suppose a moving source emits pulses of light periodically at frequency ω_0.

With what frequency does an inertial observer O see pulses?

At what time *t* does the pulse emitted at time *s* arrive at *O*?

$$t(s) = s + \frac{|\vec{r}(s)|}{c}$$

$$\dot{t}(s) = 1 + \frac{\vec{r}(s)\cdot\dot{\vec{r}}(s)}{c\,|\vec{r}(s)|} = 1 + \frac{\hat{r}(s)\cdot\dot{\vec{r}}(s)}{c}$$

$$\Delta t \cong \left(1 + \hat{r}(s)\cdot\dot{\vec{r}}(s)/c\right)\gamma(s)\,\Delta s_0$$

$$\frac{\omega_0}{\omega} = \left[\frac{1+\hat{r}\cdot\vec{v}/c}{\sqrt{1-v^2/c^2}}\right]_{\mathrm{ret}}$$

$$\ddot{t}(s) = \frac{\dot{\vec{r}}(s)\cdot\dot{\vec{r}}(s) + \vec{r}(s)\cdot\ddot{\vec{r}}(s)}{c\,|\vec{r}(s)|} - \frac{\left(\vec{r}(s)\cdot\dot{\vec{r}}(s)\right)^2}{c\,|\vec{r}(s)|^3} = \frac{\hat{r}(s)\cdot\ddot{\vec{r}}(s)}{c} + \frac{\dot{\vec{r}}(s)\cdot\left[I - \hat{r}(s)\,\hat{r}(s)\right]\cdot\dot{\vec{r}}(s)}{c\,|\vec{r}(s)|}$$