Physics 6938 | Florida Atlantic University
General Relativity | Fall, 2007

Differential Geometry Exercises I: Tensors

Due: Tuesday, 9 October 2007
Target Date: Thursday, 6 September 2007

Suggested Reading: *d'Inverno* 5.1 – 5.9, *Schutz* 2.1 – 2.31.

1. [*d'Inverno* 5.2] Write down the change of coordinates from Cartesian coordinates $(x^a) = (x, y, z)$ to spherical polar coordinates $(x'^a) = (r, \theta, \phi)$ in $\mathbb{R}^3$. Obtain the transformation matrices

$$\frac{\partial x^a}{\partial x'^b} \quad \text{and} \quad \frac{\partial x'^a}{\partial x^b}$$

expressing them both in terms of the primed coordinates. Obtain the Jacobians J and J'. Where is J' zero or infinite?

2. [*d'Inverno* 5.6] Write down the change of coordinates from Cartesian coordinates $(x^a) = (x, y)$ to plane polar coordinates $(x'^a) = (R, \phi)$ in $\mathbb{R}^2$ and obtain the transformation matrix

$$\frac{\partial x'^a}{\partial x^b}$$

expressed as a function of the primed coordinates. Find the components of the tangent vector to the curve consisting of a circle of radius a centered at the origin with the standard parameterization (see Exercise 5.1(i)) and use (5.16) to find its components in the primed coordinate system.

3. [*d'Inverno* 5.9] Show, by differentiating (5.20) with respect to x'^c, that

$$\frac{\partial^2 \phi}{\partial x^a \, \partial x^b}$$

is not a tensor.

4. [*d'Inverno* 5.11]

   a. Show that the fact that a covariant second rank tensor is symmetric in one coordinate system is a tensorial property.
   b. If X^{ab} is anti-symmetric and Y_{ab} is symmetric then prove that $X^{ab}\, Y_{ab} = 0$.

5. [*d'Inverno* 5.14] Evaluate δ^a_a and $\delta^a_b\, \delta^b_a$ in n dimensions.

6. [*d'Inverno* 5.15] Check that the definition of the Lie bracket leads to the results (5.37), (5.38), and (5.39).

7. [*d'Inverno* 5.16] In $\mathbb{R}^2$, let $(x^a) = (x, y)$ denote Cartesian and $(x'^a) = (R, \phi)$ plane polar coordinates (see Exercise 5.6).

   a. If the vector field X has components $X^a = (1, 0)$, then find X'^a.
   b. The operator grad can be written in each coordinate system as

   $$\operatorname{grad} f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} = \frac{\partial f}{\partial R}\,\hat{\mathbf{R}} + \frac{\partial f}{\partial \phi}\,\frac{\hat{\boldsymbol{\phi}}}{R},$$

   where f is an arbitrary function and

   $$\hat{\mathbf{R}} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}, \quad \hat{\boldsymbol{\phi}} = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}.$$

   Take the scalar product of $\operatorname{grad} f$ with $\mathbf{i}$, $\mathbf{j}$, $\hat{\mathbf{R}}$, and $\hat{\boldsymbol{\phi}}$ in turn to find relationships between the operators $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial R$, and $\partial/\partial \phi$.
   c. Express the vector field X as an operator in each coordinate system. Use part (b) to show that these expressions are the same.
   d. If $Y^a = (0, 1)$ and $Z^a = (-y, x)$, then find Y'^a, Z'^a, Y, and Z.
   e. Evaluate all the Lie brackets of X, Y and Z.

8. [*Schutz* 2.5 and 2.6, p. 59]

   a. Prove that a general $\binom{2}{0}$ tensor *cannot* be expressed as a simple outer product of two vectors.
   (*Hint*: count the number of components a $\binom{2}{0}$ tensor may have.)
   b. Prove that the $\binom{1}{1}$ tensor $\bar{V} \otimes \tilde{\omega}$ has components $V^i\,\omega_j$.
   c. Prove that the set of all $\binom{2}{0}$ tensors at P is a vector space under addition defined by analogy with equation (2.16b). Show that $\bar{e}_i \otimes \bar{e}_j$ is a basis for that space. (Thus, although a general $\binom{2}{0}$ tensor is not a simple outer product, it *can* be represented as a sum of such tensors.) This vector space is called $T_P \otimes T_P$.

9. [*Schutz* 2.8, p. 60] Let $\mathbf{A}$ and $\mathbf{B}$ be two $\binom{1}{1}$ tensors, and regard them as vector-valued linear functions of vectors: if $\bar{V}$ is a vector then $\mathbf{A}(\bar{V})$ and $\mathbf{B}(\bar{V})$ are vectors. Show that if we define $\mathbf{C}(\bar{V})$ to be

   $$\mathbf{C}(\bar{V}) = \mathbf{B}(\mathbf{A}(\bar{V})),$$

   then $\mathbf{C}$ is a $\binom{1}{1}$ tensor as well. Show that its components are

   $$C^i{}_j = B^i{}_k\,A^k{}_j.$$

   Discuss the relation of this with the linear transformation defined in §1.6.

10. [*Schutz* 2.13, p. 67]

    a. Show that $\{g^{ij}\}$ are the components of a $\binom{2}{0}$ tensor g^{-1}, either by showing that they transform properly, or that they define a bilinear function of one-forms.
    b. Show that if a vector basis $\{\bar{e}_i\}$ is orthonormal, so is its dual one-form basis $\{\tilde{\omega}^i\}$, in the sense that $\mathrm{g}^{-1}(\tilde{\omega}^i, \tilde{\omega}^j) = \pm\delta^{ij}$.

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Differential Geometry Exercises II: Manifolds

Due: Tuesday, 9 October 2007
Target Date: Thursday, 13 September 2007

Suggested Reading: *d'Inverno* 6.1 – 6.2; *Schutz* 2.1 – 2.4 and 3.1 – 3.5.

1. [*Schutz* 3.1, p. 78]

   a. Show that, on functions and fields,

   $$[\mathscr{L}_{\bar V}, \mathscr{L}_{\bar W}] = \mathscr{L}_{[\bar V, \bar W]}$$

   for any two twice-differentiable vector fields $\bar V$ and $\bar W$.

   b. Prove the Jacobi identity for Lie derivatives on functions and vector fields:

   $$[[\mathscr{L}_{\bar X}, \mathscr{L}_{\bar Y}], \mathscr{L}_{\bar Z}] + [[\mathscr{L}_{\bar Y}, \mathscr{L}_{\bar Z}], \mathscr{L}_{\bar X}] + [[\mathscr{L}_{\bar Z}, \mathscr{L}_{\bar X}], \mathscr{L}_{\bar Y}] = 0,$$

   where $\bar X$, $\bar Y$, $\bar Z$ are any three-times-differentiable vector fields.

   Hint: For (a) on vectors, show that (3.8) is equivalent to (2.14). For (b) on vectors, use (3.8) and the fact that, as is obvious from its definition, $\mathscr{L}_{\bar A} + \mathscr{L}_{\bar B} = \mathscr{L}_{\bar A + \bar B}$.

2. [*Schutz* 3.2 and 3.3, p. 78]

   a. Deduce the Leibniz rule

   $$\mathscr{L}_{\bar V}\,(f\bar U) = (\mathscr{L}_{\bar V}\, f)\,\bar U + f\,\mathscr{L}_{\bar V}\,\bar U$$

   from the definitions of $\mathscr{L}_{\bar V}$ on functions and vector fields.

   b. From (2.7) we know that the components of $\mathscr{L}_{\bar V}\,\bar U$ on a *coordinate basis* are

   $$(\mathscr{L}_{\bar V}\,\bar U)^i = V^j\,\frac{\partial}{\partial x^j}\,U^i - U^j\,\frac{\partial}{\partial x^j}\,V^i.$$

   Given an arbitrary basis $\{\bar e_i\}$ for vector fields, show from (a) that

   $$(\mathscr{L}_{\bar V}\,\bar U)^i = V^j\,\bar e_j(U^i) - U^j\,\bar e_j(V^i) + V^j\,U^k\,(\mathscr{L}_{\bar e_j}\bar e_k)^i,$$

   where $\bar e_j(U^i)$ means the derivative of the function U^i with respect to the vector field $\bar e_j$.

   c. Show that if one chooses a coordinate system in which $\bar V$ is a coordinate basis vector, say $\partial/\partial x^1$, then for any vector field $\bar W$

   $$(\mathscr{L}_{\bar V}\,\bar W)^i = \frac{\partial W^i}{\partial x^1}.$$

   That is, the Lie derivative is the coordinate-independent form of the partial derivative.

3. [*Schutz* 3.4, p. 79] From (3.13) and the expression (2.7) for the components of $\mathscr{L}_{\bar V}\,\bar W = [\bar V, \bar W]$, deduce that $\mathscr{L}_{\bar V}\,\tilde\omega$ has components, on a coordinate basis,

   $$(\mathscr{L}_{\bar V}\,\tilde\omega)_i = V^j\,\frac{\partial}{\partial x^j}\,\omega_i + \omega_j\,\frac{\partial}{\partial x^i}\,V^j.$$

4. [*d'Inverno* 6.2] Use (6.17) to find expressions for $\mathscr{L}_X Z_{bc}$ and $\mathscr{L}_X (Y^a Z_{bc})$. Use these expressions and (6.15) to check the Leibniz property in the form (6.12).

5. Let (x, y, z) denote a point in $\mathbb{R}^3$ with $x^2 + y^2 + z^2 = 1$. Define the maps

$$\psi_N(x, y, z) := \left(\frac{x}{1+z}, \frac{y}{1+z}\right) \quad \text{and} \quad \psi_S(x, y, z) := \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

from S^2 to $\mathbb{R}^2$.

   a. Show that ψ_N is defined everywhere on S^2 except at the south pole $(0, 0, -1)$ and that ψ_S is defined everywhere except at the north pole $(0, 0, 1)$.
   b. Calculate the coordinate transformation $\psi_N \circ \psi_S^{-1}(u, v)$ on the largest subset of $\mathbb{R}^2$ for which it can be defined. Show that this mapping from $\mathbb{R}^2$ to itself is smooth everywhere it is defined.
   c. Conclude that S^2 is a two-dimensional real manifold.

6. The *complex projective plane* $\mathbb{C}P^1$ is the set of "complex lines" in $\mathbb{C}^2$ — the set of vectors $(z^1, z^2) \in \mathbb{C}^2$ up to overall scaling $(z^1, z^2) \mapsto (\alpha z^1, \alpha z^2)$ by an arbitrary complex number α. More mathematically, $\mathbb{C}P^1$ is the set of equivalence classes $[z^1, z^2]$ of points in $\mathbb{C}^2$, where two points are equivalent if and only if they are complex scalings of one another.

   a. Define the mappings
   $$\zeta_1 := \frac{z^1}{z^2} \quad \text{and} \quad \zeta_2 := \frac{z^2}{z^1}$$
   from $\mathbb{C}^2$ to the complex plane $\mathbb{C}$. Show that each is in fact a (complex) coordinate chart mapping $\mathbb{C}P^1$ to $\mathbb{C}$.
   b. What are the repsective domains $U_{1,2}$ of the charts $\zeta_{1,2}$? That is, on what set of "lines" $[z^1, z^2] \in \mathbb{C}P^1$ is each coordinate $\zeta_{1,2}$ well-defined?
   c. Show that the inverse charts mapping $w \in \mathbb{C}$ back to $\mathbb{C}P^1$ can be written
   $$\zeta_1^{-1}(w) = [w, 1] \quad \text{and} \quad \zeta_2^{-1}(w) = [1, w].$$
   d. Find the coordinate transformation mapping the region $\zeta_1(U_1 \cap U_2)$ of the complex plane to the region $\zeta_2(U_1 \cap U_2)$ of the complex plane. Show that it is smooth and invertible throughout $\zeta_1(U_1 \cap U_2)$, and that its inverse is also smooth. (In fact, it is analytic in the complex-variables sense).
   e. Conclude that $\mathbb{C}P^1$ is a one-dimensional *complex* manifold.

7. Define the mapping
$$\phi(x, y, z) := [1 + z, x + \mathrm{i}y] = [x - \mathrm{i}y, 1 - z]$$
from S^2 to $\mathbb{C}P^1$.

   a. Prove the second equality in the above definition of $\phi(x, y, z)$.
   b. Show that this mapping is invertible. That is, find a formula giving (x, y, z) as a function of z^1 and z^2, and check that this formula is invariant under scalings $(z^1, z^2) \mapsto (\alpha z^1, \alpha z^2)$.
   c. Calculate the four mappings $\zeta_{1,2} \circ \phi \circ \psi_{N,S}^{-1}$ from $\mathbb{R}^2$ to $\mathbb{C}$. Show that each is smooth and has a smooth inverse.
   d. Conclude that, viewed as a two-dimensional *real* manifold, $\mathbb{C}P^1$ is diffeomorphic to S^2.

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Differential Geometry Exercises III: Connections

Due: Tuesday, 9 October 2007
Target Date: Tuesday, 25 September 2007

Suggested Reading: *d'Inverno* 6.3 – 6.12; *Schutz* 6.1 – 6.6.

1. [*d'Inverno* 6.5] Assuming (6.22) and (6.25), apply the Leibniz rule to the covariant derivative of $X_a\, Y^a$, where Y^a is arbitrary, to verify (6.26).

2. [*d'Inverno* 6.9] If s is an affine parameter, then show that, under the transformation

$$s \mapsto \bar{s} = \bar{s}(s),$$

the parameter $\bar{s}$ will be affine if and only if $\bar{s} = \alpha s + \beta$, where α and β are constants.

3. [*d'Inverno* 6.10] Show that $\nabla_c\, \nabla_d\, X^a{}_b - \nabla_d\, \nabla_c\, X^a{}_b = R^a{}_{ecd}\, X^e{}_b - R^e{}_{bcd}\, X^a{}_e$.

4. [*d'Inverno* 6.11] Show that $\nabla_X\, (\nabla_Y\, Z^a) - \nabla_Y\, (\nabla_X\, Z^a) - \nabla_{[X,Y]}\, Z^a = R^a{}_{bcd}\, Z^b\, X^c\, Y^d$.

5. [*d'Inverno* 6.14] The line elements of $\mathbb{R}^3$ in Cartesian, cylindrical polar and spherical polar, and spherical polar coordinates are given respectively by

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 = \mathrm{d}R^2 + R^2\, \mathrm{d}\phi^2 + \mathrm{d}z^2 = \mathrm{d}r^2 + r^2\, \mathrm{d}\theta^2 + r^2 \sin^2\theta\, \mathrm{d}\phi^2.$$

Find g_{ab}, g^{ab} and g in each case.

6. [*d'Inverno* 6.17] Find the geodesic equation for $\mathbb{R}^3$ in cylindrical polars.
Hint: Use the results of Exercise 6.14 to compute the metric connection and substitute in (6.68).

7. [*d'Inverno* 6.20] Suppose we have an arbitrary symmetric connection Γ^a_{bc} satisfying $\nabla_c\, g_{ab} = 0$. Deduce that Γ^a_{bc} must be the metric connection.
Hint: Use the equation to find expressions for $\partial_b\, g_{dc}$, $\partial_c\, g_{db}$ and $-\partial_d\, g_{bc}$, as in (6.78), add the equations together and multiply by $\frac{1}{2}\, g^{ad}$.

8. [*d'Inverno* 6.23 and 6.24]

   a. Establish the identities (6.78) and (6.79). Show that (6.78) is equivalent to $R^a{}_{[bcd]} \equiv 0$.
   b. Establish the identity (6.82). Show that (6.82) is equivalent to $R_{de[eb;c]} \equiv 0$. Deduce (6.86).

   Hint: Choose an arbitrary point P and introduce geodesic coordinates at P.

9. [*Schutz* 6.10, p. 208] Suppose a manifold has two connections defined on it, with Christoffel symbols $\Gamma^k{}_{ij}$ and $\Gamma'^k{}_{ij}$. Show that

$$D^k{}_{ij} \equiv \Gamma^k{}_{ij} - \Gamma'^k{}_{ij}$$

are the components of a $\binom{1}{2}$ tensor. Show that the tensor $\mathbf{D}$ is symmetric in its vector arguments if and only if both connections have the same torsion tensor.

10. [*Schutz* 6.11, p. 208] A manifold has a symmetric connection. Show that in any expression for the components of the Lie derivative, all commas can be replaced by semicolons. An example:

$$(\mathscr{L}_{\bar{U}}\, \tilde{\omega})_i = \omega_{i,j}\, U^j + \omega_j\, U^j{}_{,i} = \omega_{i;j}\, U^j + \omega_j\, U^j{}_{;i}.$$

(Naturally, *all* commas must be changed, not just some.)

11. [*Schutz* 6.14, p. 211] The components of the Riemann tensor $R^i{}_{jkl}$, are defined by

$$[\nabla_i, \nabla_j]\, \bar{e}_k - \nabla_{[\bar{e}_i, \bar{e}_j]}\, \bar{e}_k = R^l{}_{kij}\, \bar{e}_l.$$

(Where $\bar{e}_i$ is possibly a non-coordinate basis. — CB)

a. Show that in a *coordinate* basis

$$R^l{}_{kij} = \Gamma^l{}_{kj,i} - \Gamma^l{}_{ki,j} + \Gamma^m{}_{kj}\, \Gamma^l{}_{mi} - \Gamma^m{}_{ki}\, \Gamma^l{}_{mj}.$$

b. In a non-coordinate basis, define the commutation coefficients $C^i{}_{jk}$ by

$$[\bar{e}_j, \bar{e}_k] = C^i{}_{jk}\, \bar{e}_i.$$

Show that

$$R^l{}_{kij} = \Gamma^l{}_{kj,i} - \Gamma^l{}_{ki,j} + \Gamma^m{}_{kj}\, \Gamma^l{}_{mi} - \Gamma^m{}_{ki}\, \Gamma^l{}_{mj} - C^m{}_{ij}\, \Gamma^l{}_{km},$$

where $f_{,i} \equiv \bar{e}_i[f]$.

c. Show that

$$R^l{}_{k(ij)} \equiv \tfrac{1}{2}\, (R^l{}_{kij} + R^l{}_{kji}) = 0 \quad \text{and} \quad R^l_{[kij]} = 0.$$

Hint: For the second equality, use normal coordinates. The result, of course, is independent of the basis.

d. Using (c) show that in an n-dimensional manifold, the number of linearly independent components opf $R^l{}_{kij}$ is

$$n^4 - n^2\, \frac{n(n+1)}{2} - n\, \frac{n(n-1)(n-2)}{3!} = \tfrac{1}{3}\, n^2\, (n^2 - 1).$$

12. [*Schutz* 6.16, p. 215] Consider a two-dimensional flat space with Cartesian coordinates x, y and polar coordinates r, θ.

a. Use the fact that $\bar{e}_x$ and $\bar{e}_y$ are globally parallel vector fields ($\bar{e}_x(P)$ is parallel to $\bar{e}_x(Q)$ for arbitrary P, Q) to show that

$$\Gamma^r{}_{\theta\theta} = -r, \quad \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \frac{1}{r},$$

and all other Γ's are zero in polar coordinates.

b. For an arbitrary vector field $\bar{V}$, evaluate $\nabla_i\, V^j$ and $\nabla_i\, V^i$ for polar cooridnates in terms of the components V^r and V^θ.

c. For the basis

$$\hat{\mathbf{r}} = \frac{\partial}{\partial r}, \quad \hat{\boldsymbol{\theta}} = \frac{1}{r}\, \frac{\partial}{\partial \theta}$$

find all the Christoffel symbols.

d. Same as (b) for the basis in (c).

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Differential Geometry Exercises IV: Integration

Due: Tuesday, 9 October 2007
Target Date: Tuesday, 2 October 2007

Suggested Reading: *d'Inverno* 7.1 – 7.4; *Schutz* 4.1 – 4.23.

1. [*d'Inverno* 7.4] Show that, for any vector field T^a, the divergence theorem in four dimensions can be written in the form

$$\int_{\partial\Omega} T^a \sqrt{-g}\, \mathrm{d}S_a = \int_{\Omega} \nabla_a T^a \sqrt{-g}\, \mathrm{d}^4x.$$

2. [*Schutz* 4.8, p. 118] Show that if $\tilde{p}$ is a one-form and $\tilde{q}$ a two-form, then

$$(\tilde{p} \wedge \tilde{q})_{ijk} = p_i\, q_{jk} + p_j\, q_{ki} + p_k\, q_{ij} = 3\, p_{[i}\, q_{jk]}.$$

   More generally, show that if $\tilde{p}$ is a p-form and $\tilde{q}$ a q-form,

$$(\tilde{p} \wedge \tilde{q})_{i\ldots jk\ldots l} = C^{p+q}_p\, p_{[i\ldots j}\, q_{k\ldots l]}.$$

   (The symbol C^{p+q}_p here is $\binom{p+q}{p}$ from the binomial theorem. — CB)

3. [*Schutz* 4.9, p. 120] Prove (4.16).

4. [*Schutz* 4.12 and 4.13, pp. 131-132]

   a. Show that the determinant of an $n \times n$ matrix with elements A^{ij} $(i, j = 1, \ldots, n)$ is

$$\det A = \epsilon_{ij\ldots k}\, A^{1i}\, A^{2j} \ldots A^{nk}.$$

   Hint: The determinant of an $n \times n$ matrix is defined in terms of $(n-1) \times (n-1)$ determinants by the cofactor rule. Use that rule to prove this results by induction from the 2×2 case.

   b. Show that

$$\det A = \frac{1}{n!}\, \epsilon_{ab\ldots c}\, \epsilon_{ij\ldots k}\, A^{ai}\, A^{bj} \ldots A^{ck}.$$

   c. If a manifold has a metric, let $\{\tilde{\omega}^i\}$ be an orthonormal basis for one-forms, and define $\tilde{\omega}$ to be the preferred volume-form

$$\tilde{\omega} = \tilde{\omega}^1 \wedge \tilde{\omega}^2 \wedge \ldots \wedge \tilde{\omega}^n.$$

   Show that, if $x^{k'}$ is an arbitrary coordinate system,

$$\tilde{\omega} = |g|^{1/2}\, \tilde{\mathrm{d}}x^{1'} \wedge \tilde{\mathrm{d}}x^{2'} \wedge \ldots \wedge \tilde{\mathrm{d}}x^{n'},$$

   where g is the determinant of the matrix of components $g_{i'j'}$ of the metric tensor in these coordinates.

5. [*Schutz* 4.14, p. 135]

   a. Show that
   $$\tilde{\mathrm{d}}(f\,\tilde{\mathrm{d}}g) = \tilde{\mathrm{d}}f \wedge \tilde{\mathrm{d}}g.$$

   b. Use (a) to show that if
   $$\tilde{\alpha} = \frac{1}{p!}\,\alpha_{i\ldots j}\,\tilde{\mathrm{d}}x^i \wedge \ldots \wedge \tilde{\mathrm{d}}x^j$$
   is the expansion for the p-form $\tilde{\alpha}$ in a coordinate basis, then
   $$\tilde{\mathrm{d}}\tilde{\alpha} = \frac{1}{p!}\,\frac{\partial \alpha_{i\ldots j}}{\partial x^k}\,\tilde{\mathrm{d}}x^k \wedge \tilde{\mathrm{d}}x^i \wedge \ldots \wedge \tilde{\mathrm{d}}x^j,$$
   and hence that $(\tilde{\mathrm{d}}\tilde{\alpha})_{ki\ldots j} = (p+1)\,\partial_{[k}\,\alpha_{i\ldots j]}$.

6. [*Schutz* 4.16, p. 137] Use (4.50), (4.52), and property (iii) of §4.14 to show that (in three-dimensional Euclidean vector calculus) the divergence of a curl and the curl of a gradient both vanish.

7. [*Schutz* 4.18, p. 142] Use the local exactness theorem to show that locally (in three-dimensional Euclidean vector calculus) a curl-free vector field is a gradient and a divergence-free vector field is a curl.

8. [*Schutz* 4.20 and 4.21, p. 148]

   a. From (4.77) show that, if coordinates are chosen in which $\tilde{\omega} = f\,\tilde{\mathrm{d}}x^1 \wedge \ldots \wedge \tilde{\mathrm{d}}x^n$, then
   $$\mathrm{div}_{\tilde{\omega}}\,\bar{\xi} = \frac{1}{f}\,(f\,\xi^i)_{,i}.$$

   b. In Euclidean three-space the preferred volume three-form is $\tilde{\omega} = \tilde{\mathrm{d}}x \wedge \tilde{\mathrm{d}}y \wedge \tilde{\mathrm{d}}z$. Show that in spherical polar coordinates this is $\tilde{\omega} = r^2 \sin\theta\,\tilde{\mathrm{d}}r \wedge \tilde{\mathrm{d}}\theta \wedge \tilde{\mathrm{d}}\phi$. Use (4.80) to show that the divergence of a vector $\bar{\xi} = \xi^r\,\partial_r + \xi^\theta\,\partial_\theta + \xi^\phi\,\partial_\phi$ is
   $$\mathrm{div}\,\bar{\xi} = \frac{1}{r^2}\,\frac{\partial}{\partial r}(r^2\,\xi^r) + \frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\,(\sin\theta\,\xi^\theta) + \frac{\partial \xi^\phi}{\partial\phi}.$$

9. [Schutz 4.23, p. 149]

   a. Show from (4.77) that another expression for the divergence of a vector $\bar{\xi}$ is
   $$\mathrm{div}_{\tilde{\omega}}\,\bar{\xi} = {*\mathrm{d}*}\,\bar{\xi},$$
   where the $*$-operation is the dual with respect to $\tilde{\omega}$ introduced earlier.

   b. For any p-vector $\mathbf{F}$ define
   $$\mathrm{div}_{\tilde{\omega}}\,\mathbf{F} = (-1)^{n(p-1)}\,{*\mathrm{d}*}\,\mathbf{F}.$$
   Show that $\mathrm{div}_{\tilde{\omega}}\,\mathbf{F}$ is a $(p-1)$-vector. Show that if $\tilde{\omega}$ has components $\epsilon_{i\ldots j}$ in some coordinate system, then
   $$(\mathrm{div}_{\tilde{\omega}}\,\mathbf{F})^{i\ldots j} = F^{ki\ldots j}{}_{,k}$$
   in those coordinates.

   c. Generalize part (a) of the previous exercise to p-vectors.

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Differential Geometry Exercises V: Symmetry

Due: Tuesday, 9 October 2007
Target Date: Tuesday, 9 October 2007

Suggested Reading: *d'Inverno* 7.5 – 7.7; *Schutz* 3.6 – 3.13 and 5.11 – 5.14.

1. [*d'Inverno* 7.7] Use (7.45), (7.46), and (7.47) to find the geodesic equations of the spherically symmetric line element given in Exercise 6.31. Use the equations to read off directly the components Γ^a_{bc} and check them with those obtained in Exercise 6.31(ii).
   Hint: Remember $\Gamma^a_{bc} = \Gamma^a_{cb}$.

2. [*d'Inverno* 7.12] Consider the following operator identity:

$$\mathscr{L}_U \, \mathscr{L}_V - \mathscr{L}_V \, \mathscr{L}_U = \mathscr{L}_{[U,V]}.$$

   (U and V are vector fields. — CB)

   a. Check it holds when applied to an arbitrary scalar function f.
   b. Check it holds when applied to an arbitrary contravariant vector field m^a.
      Hint: Use the Jacobi identity.
   c. Deduce that the identity holds when applied to a covariant vector field p_a.
      Hint: Let $f = m^a \, p_a$, where m^a is arbitrary.
   d. Use the identity to prove that if U and V a Killing vector fields, then so is their commutator $[U, V]$.
   e. Given that ∂_x and $-y \, \partial_x + x \, \partial_y$ are Killing vector fields, find another.

3. [*d'Inverno* 7.13 and 7.14]

   a. By making use of the identity

$$R^a{}_{bcd} + R^a{}_{cdb} + R^a{}_{dbc} = 0$$

   or otherwise, prove that a Killing vector satisfies

$$\nabla_c \, \nabla_b \, X_a = R_{abcd} \, X^d.$$

   b. Use this result to prove that any Killing vector satisfies

$$g^{bc} \, \nabla_b \, \nabla_c \, X_a - R_{ab} \, X^b = 0.$$

4. [*Schutz* 6.20, p. 216] Show that for an arbitrary vector $\bar{V}$

$$(\mathscr{L}_{\bar{V}} \, \mathrm{g})_{ij} = \nabla_i \, V_j + \nabla_j \, V_i.$$

   Therefore a Killing vector obeys *Killing's equation* $\nabla_{(i} \, V_{j)} = 0$.

5. [*Schutz* 3.5, p. 81]

   a. Show that if $\bar{V}$ and $\bar{W}$ are linear combinations (not necessarily with constant coefficients) of m vector fields that all commute with one another, then the Lie bracket of $\bar{V}$ and $\bar{W}$ is a linear combination of the same m fields.

   b. Prove the same result when the m vector fields have Lie brackets which are nonvanishing linear combinations of the m fields.

6. Let ξ^a be a Killing vector field for the metric g_{ab}, and let η^a be the tangent vector to a geodesic of g_{ab} in an affine parameterization. Show that the inner product of these vectors is constant along the geodesic. What happens to this conserved quantity if one changes affine parameterizations? What happens if the parameterization is not affine?

7. Let g_{ab} be a *stationary* spacetime metric — meaning that it has a time-like Killing field t^a — that solves the vacuum Einstein equations $R_{ab} = 0$.

   a. Show that $F_{ab} := \nabla_a t_b$ satisfies the source-free Maxwell equations.

   b. Suppose that t^a is in fact a *static* Killing field — meaning that it is orthogonal everywhere to some space-like surfaces Σ. Calculate the electric and magnetic parts of F_{ab} on the static slices Σ.

   c. Use the Gauss law to compute the "electric charge" of the *Schwarzschild metric*

   $$\mathrm{d}s^2 = (1 - 2M/r)\,\mathrm{d}t^2 + (1 - 2M/r)^{-1}\,\mathrm{d}r^2 + r^2\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2).$$

   Note that the static Killing field in this case is ∂_t, and that the static slices are the surfaces of constant t in spacetime.
   Hint: The electric charge is given by a flux integral. Show that one gets the same result no matter which two-sphere one uses in the integral. Then, calculate in the asymptotic region where $r \to \infty$.