Physics 6938
General Relativity

Florida Atlantic University
Fall 2007

Final Exam

Due: Tuesday, December 11, 2007

- **Write your name on this cover page.**
- Exams are due by 6:00 PM on Tuesday.
- As with the homework, you may discuss these problems with other students in the class. But copying is strictly forbidden.
- Please contact me with any questions or concerns.
- Good luck!

1. A rocket of initial rest-mass m_0 accelerates along a straight line by ejecting material at a speed u, as measured in the instantaneously co-moving inertial frame, in the opposite direction.

   a. Fix an inertial frame in which the initial velocity v_0 of the rocket vanishes. Show that, when the speed of the rocket relative to that frame is v, its rest mass m is given by

   $$m = \left(\frac{c-v}{c+v}\right)^{c/2u} m_0.$$

   b. Derive the non-relativistic result

   $$m = \mathrm{e}^{-v/u}\, m_0$$

   for this problem, and show that the relativistic result reduces to this in the limit where $u/c \to 0$.

   c. Discuss and contrast the conservation principles that hold in the relativistic and non-relativistic cases, respectively.

2. Tom (Major, USAF) is an unfortunate astronaut. He and his spacecraft have just passed through the event horizon of a Schwarzschild black hole with mass M.

   a. Show that Tom will live no longer than $\tau = \pi M$ longer, as measured on his own clock, no matter how he subsequently fires his thrusters.
   b. Can Tom see the singularity that spells his doom? Does he know that he has passed through the event horizon? Can he tell Ground Control whose shirts he wears?
   c. Can you hear me, Major Tom?
   d. Can you hear me, Major Tom?

3. Consider the metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}z^2 + X^2(u)\,\mathrm{d}x^2 + Y^2(u)\,\mathrm{d}y^2$$

where $u := t - z$ is a null coordinate and $X(u)$ and $Y(u)$ are arbitrary functions.

a. Show that the vacuum Einstein equations for this metric are equivalent to

$$\frac{X''(u)}{X(u)} + \frac{Y''(u)}{Y(u)} = 0,$$

where primes denote derivatives with respect to u.

b. How many independent Killing vectors does this metric have? Argue that it may be interpreted as a *non-perturbative* plane gravitational wave propagating in the $+z$-direction.

4. A light ray with "impact parameter at infinity" $b := L/E$ moves in the gravitational field of a Schwarzschild black hole of mass M. Here,

$$E := \left(1 - \frac{2M}{r}\right)\dot{t} \quad \text{and} \quad L := r^2 \sin^2\theta\,\dot{\phi}$$

are the quantities defined in class that are conserved along a geodesic.

a. Show that the point of closest approach (smallest r) to the black hole along the ray's null geodetic trajectory occurs at the radius r_0 given by

$$\frac{\sqrt{3}\,r_0}{2b} = \cos\left[\frac{1}{3}\cos^{-1}\left(-3\,\frac{\sqrt{3}\,R}{2b}\right)\right],$$

where $R := 2M$ denotes the Schwarzschild radius of the horizon.
Hint: The elementary trigonometric triple-angle formulae

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \qquad \text{and} \qquad \sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

will help enormously to solve the cubic polynomial you should find. Only the largest root is physical.

b. The right side of the result in part (a) is real only if $b^2 > 27R^2/4$. If this condition is violated, show that the only real r where $\dot{r} = 0$ is negative, and therefore that all three are unphysical. That is, there is no turning point if the impact parameter is too small.
Hint: Here, the formula

$$\cosh 3u = 4\cosh^3 u - 3\cosh u$$

from the hyperbolic trigonometry will help.

c. Describe *qualitatively* what happens in the limit $b^2 \to 27R^2/4$ where the impact parameter approaches its critical value from above.
Hint: What is the turning radius r_0 in this limit?

d. **(Optional)** Write down the integral for the total change in the angular coordinate ϕ as the photon in this problem scatters off the black hole and back out to infinity. Show that this integral diverges in the critical limit of part (c), and thereby derive the absorption cross section for radiation of the Schwarzschild black hole.